Lecture 1: ILP Formulations for the Graph Coloring Problem

Part 1.2: Graph Coloring

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Computer Science

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Outline

1. **Graph Coloring**
   - Definitions and Properties
   - Applications
   - Exact approaches for the graph coloring problem
2. **ILP Formulations**
   - Assignment Model
   - Representatives Formulations
   - Partial Ordering based Models
   - Set Covering based Model
3. **Computational Studies and Results**
4. **Conclusion and Open Questions**

Graph Coloring Literature: Surveys

DISCLAIMER: There is a vast amount of literature concerning the graph coloring problem. Here, we can only discuss a tiny part of it!

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- T. Husfeldt: Graph colouring algorithms, Cambridge University Press, 2015
- R.M.R. Lewis: Guide to graph colouring: Algorithms and applications, Guide to Graph Colouring, 2016
- A.M.D. Lima and R. Carmo: Exact algorithms for the graph colouring problem. Revista de Informática Teórica e Aplicada, 2018
- E. Malaguti, and P. Toth: A survey on vertex coloring problems. International Transactions in Operational Research 2010

Graph Coloring Literature: Explicit ILP Models

- I. Mendez-Diaz and P. Zabala: A branch-and-cut algorithm for graph coloring, Discrete Applied Mathematics 2006
- M. Campêlo, M., R. Corrêa, R. and Frota, Y., Cliques, holes and the vertex coloring polytope, Information Processing Letters 2004
- M. Campêlo, V.A. Campos, and R.C. Corrêa: On the asymmetric representatives formulation for the vertex coloring problem, Discrete Applied Mathematics, 2008
- A. Jabrayilov and P. Mutzel: New integer linear models for the vertex coloring problem, LATIN 2018
- A. Jabrayilov and P. Mutzel: Strengthened partial-ordering based ILP models for the vertex coloring problem, arXiv 2022

Graph Coloring Literature: Independent Set based ILP Models

- A. Mehrotra and M.A. Trick: A column generation approach for graph coloring, INFORMS Journal on Computing 1996
- E. Malaguti, M. Monaci, and P. Toth: An exact approach for the vertex coloring problem, Discrete Optimization 2011
- S. Held, W. Cook, and E.C. Sewell: Maximum-weight stable sets and safe lower bounds for graph coloring, IPCO 2011
- S. Held, W. Cook, and E.C. Sewell: Safe lower bounds for graph coloring, Mathematical Programming Computation 2012
- W.-J. van Hoeve: Graph coloring with decision diagrams, Mathematical Programming, 2022

Some Graph Coloring Instances and Results

- D. Johnson, A. Mehrotra, and M. Trick: DIMACS graph coloring instances 2002, https://mat.tepper.cmu.edu/COLOR02/
- S. Gualandi and M. Chiarandini: Graph coloring instances 2017, https://sites.google.com/site/graphcoloring/vertex-coloring
- N. Tamura: CSP2SAT: GCP benchmark results, https://cspsat.gitlab.io/csp2sat/gcp/

Overview

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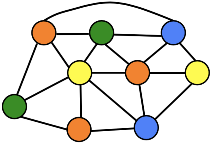
Graph Coloring Problem



Definitions

Let $G = (V, E)$ be an undirected graph.

- A (vertex-) (coloring) of G is an assignment of one color to each vertex, so that adjacent vertices are colored differently.
- The smallest number of colors for a given graph G is called the chromatic number of G: $\chi(G)$.

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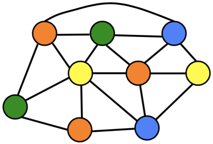

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The coloring problem: "Does G have a coloring with at most k colors?" (in short: "Is G k-colorable?") is NP-complete [Karp 1972].

Properties

Notations

- Let $\omega(G)$ denote the clique number (size of largest clique),
- $\Delta(G)$ the largest vertex degree in G

The following holds

- $\chi(G) = 1 \Leftrightarrow$

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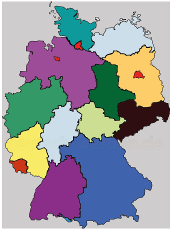
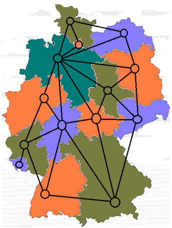
- $\chi(G) \geq 3 \Leftrightarrow G$ has an odd cycle
- $\chi(G) \geq \omega(G)$
- $\chi(G) \leq \Delta(G) + 1$
- $\chi(G) \leq \Delta(G) \Leftrightarrow$ (G is not complete) and (G is not an odd cycle)

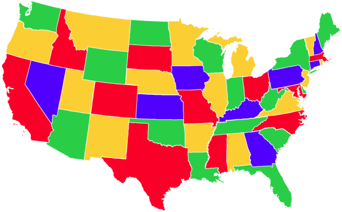
Four Color Theorem

Four Color Conjecture by Francis Guthrie (1852)

- Four colors are sufficient to color a map so that no two adjacent regions receive the same color.







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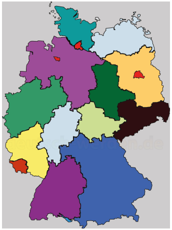

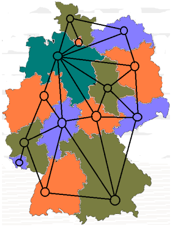

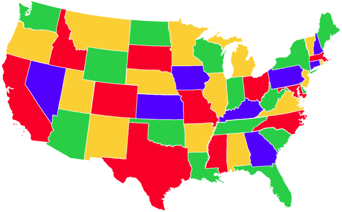

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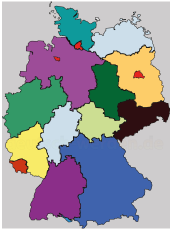

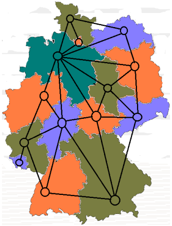

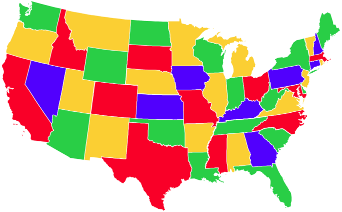

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$\Rightarrow$ first generally accepted major computer-aided proof

Image sources: https://de.wikiversity.org/, https://geoawesomeness.com

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Remarks

- this was the first generally accepted major computer-aided proof
- Robertson, Sanders, Seymour, and Thomas (1997) found a simpler proof (also computer-aided)
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It is NP-complete to decide if a given planar graph is 3-colorable.

Strong Perfect Graph Theorem

Definitions

- A perfect graph is a graph G in which, for every induced subgraph F of G: $\omega(F) = \chi(F)$.
- G is a Berge graph if no induced subgraph of G is an odd cycle of length at least 5 or the complement of one.

Perfect graphs include, e.g., bipartite graphs, chordal graphs

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Strong Perfect Graph Conjecture by Claude Berge (1961)

- G is perfect $\Leftrightarrow$ G is Berge.

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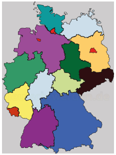
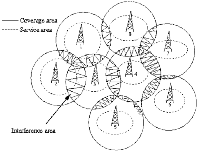
Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour, Thomas (2002)

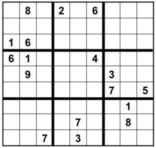
- Strong perfect graph conjecture holds.
- Every Berge graph either falls into one of five basic classes, or it has one of four different types of structural types of decomposition into simpler graphs (stronger conjecture by Conforti, Cornuéjols, and Vuskovic and Cornuéjols, Robertson, Seymour, and Thomas).

Overview

Applications







- map coloring
- frequency assignment of telecommunication networks
- computer register allocation problem

Image sources: Idoumghar, Schott, IEEEE Transactions on Broadcasting 2009

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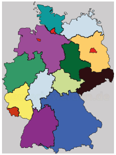

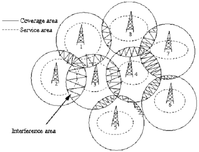

	8		2		6			
1	6							
6	1				4			
	9					3		
						7		5
							1	
				7			8	
		7		3				

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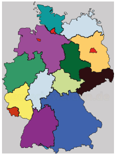

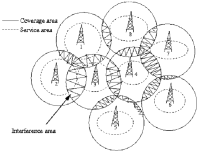

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Sudoku

Sudoku Task

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Sudoku

Sudoku Task

Complete a given 9x9 matrix, so that in each row, in each column, and in each of the marked 3x3 squares each of the numbers $1, 2, \ldots, 9$ occurs exactly once.

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A (correct) Sudoku has exactly one feasible solution.

Sudoku: Formulation as graph coloring problem

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The vertices of the graph correspond to the cells. For each pair of vertices within the same row, the same column, or the same marked square there is an edge.

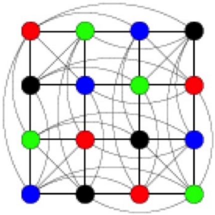

Abbildung: small example with 2x2 squares

Coloring task: Find a coloring with 9 colors (here: 4 colors) that are consistent with the given partial coloring.

Overview

Exact approaches for the graph coloring problem

- Dynamic programming approaches
- Branch-and-bound based enumeration
- SAT based approaches
- ILP approaches

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In the following: Let G be a connected graph with at least 3 vertices and H be an upper bound of $\chi(G)$.

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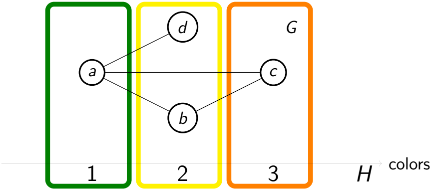
Assignment-based ILP model

Binary variables

- For each vertex $v \in V$ and color $i \in 1, \ldots, H$ we introduce assignment variables $x_{v,i} = \begin{cases} 1 & v \text{ is assigned to color } i \\ 0 & \text{otherwise.} \end{cases}$



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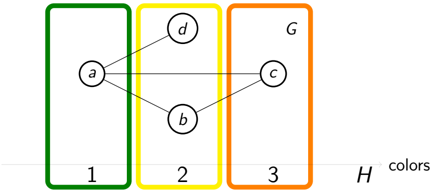
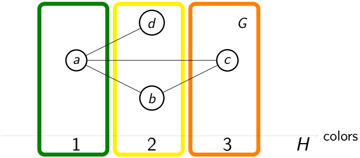
- For each color $i \in 1, \ldots, H$ we introduce $w_i = \begin{cases} 1 & \text{color } i \text{ is used} \\ 0 & \text{otherwise} \end{cases}$



Assignment model



$$\min \sum_{1\leq i\leq H} w_i$$

$$s.t. \quad \sum_{i=1}^{H} x_{v,i} = 1 \qquad \forall v \in V \qquad (1)$$

$$x_{u,i} + x_{v,i} \leq w_i \qquad \forall (u,v) \in E,\ i = 1,\ldots,H \qquad (2)$$

$$x_{v,i}, w_i \in \{0,1\} \qquad \forall v \in V,\ i = 1,\ldots,H \qquad (3)$$

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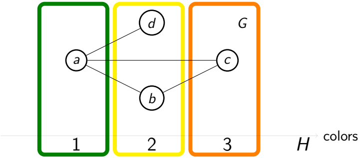

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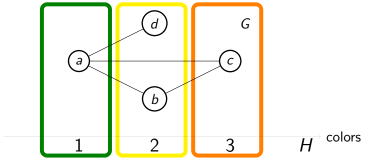
- number of variables: $H + H|V|$ and constraints: $|V| + H|E|$

Assignment model



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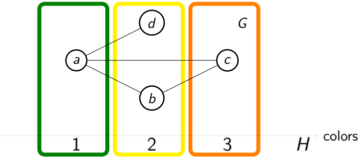

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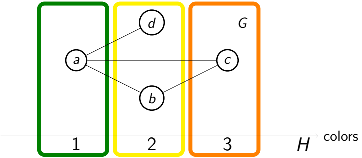
- number of variables: $H + H|V|$ and constraints: $|V| + H|E|$
- simple and easy to use, easily adaptable for problem variations
- problem: many optimal solutions that are symmetric to each other
- $\Rightarrow$ problem for branch-and-bound approaches

Assignment model (extended)

In order to overcome the symmetry, Mendez-Diaz and Zabala (2006) have suggested to add the following constraints:



$$\min \sum_{1 \le i \le H} w_i$$

$$\text{s.t.} \quad \sum_{i=1}^{H} x_{v,i} = 1 \qquad \forall v \in V \qquad (4)$$

$$x_{u,i} + x_{v,i} \le w_i \qquad \forall (u,v) \in E,\ i = 1, \ldots, H \qquad (5)$$

$$w_i \le \sum_{v \in V} x_{vi} \qquad i = 1, \ldots, H \qquad (6)$$

$$w_i \le w_{i-1} \qquad i = 2, \ldots, H \qquad (7)$$

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Representatives Formulation

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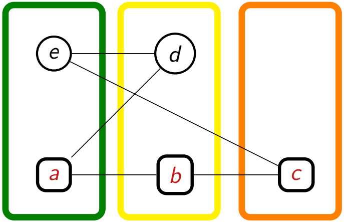
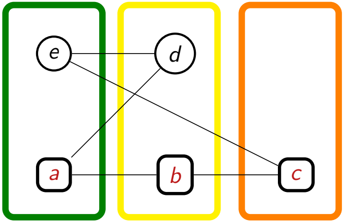


Representatives Formulation

Binary variables

- For each vertex $u \in V$ we introduce a variable
  $x_{uu} = \begin{cases} 1 & u \text{ is a representative} \\ 0 & \text{otherwise.} \end{cases}$

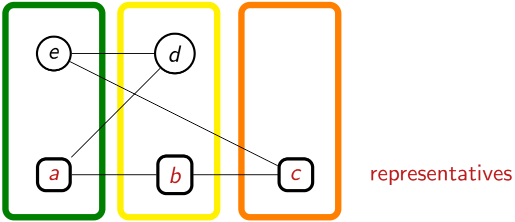


representatives

Representatives Formulation

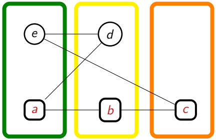
Binary variables

- For each vertex $u \in V$ we introduce a variable
$$x_{uu} = \begin{cases} 1 & u \text{ is a representative} \\ 0 & \text{otherwise.} \end{cases}$$
- For each pair of non-adjacent vertices $u, v \in V$ we introduce
$$x_{uv} = \begin{cases} 1 & u \text{ represents the color of } v \\ 0 & \text{otherwise.} \end{cases}$$



- For example: $x_{a,e} = 1$, $x_{c,c} = 1$, $x_{a,b} = 0$, ...

Representatives Formulation

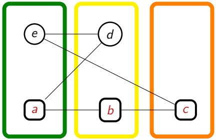


$$
\begin{aligned}
\min \quad & \sum_{u \in V} x_{uu} & & \\
s.t. \quad & \sum_{u \in \bar{N}(v) \cup v} x_{uv} \geq 1 & \forall v \in V & \quad (9)\\
& x_{uv} + x_{uw} \leq x_{uu} & \forall u \in V,\ \forall (v,w) \in E \text{ and } v,w \in \bar{N}(u) & \quad (10)\\
& x_{uu} \in \{0,1\} & \forall u \in V & \quad (11)\\
& x_{uv} \in \{0,1\} & \forall \text{ non-adjacent pairs } u,v. & \quad (12)
\end{aligned}
$$

- $\bar{N}(v)$ is the set of vertices not adjacent to v

Campêlo, Corrêa and Frota 2004, Campêlo, Campos and Corrêa 2006

Representatives Formulation



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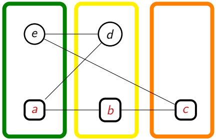
- $\bar{N}(v)$ is the set of vertices not adjacent to v
- number of variables: $|(V \times V) \setminus E|$ and constraints: up to $|V| + |V||E|$

Campêlo, Corrêa and Frota 2004, Campêlo, Campos and Corrêa 2006

Representatives Formulation



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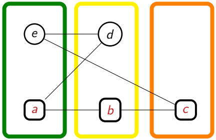
\end{aligned}
$$

- $\bar{N}(v)$ is the set of vertices not adjacent to v
- number of variables: $|(V \times V) \setminus E|$ and constraints: up to $|V| + |V||E|$
- simple and compact

Campêlo, Corrêa and Frota 2004, Campêlo, Campos and Corrêa 2006

Representatives Formulation



$$\min \sum_{u \in V} x_{uu}$$

$$\text{s.t.} \sum_{u \in \bar{N}(v) \cup v} x_{uv} \geq 1 \qquad \forall v \in V \tag{9}$$

$$x_{uv} + x_{uw} \leq x_{uu} \qquad \forall u \in V,\ \forall (v,w) \in E \text{ and } v,w \in \bar{N}(u) \tag{10}$$

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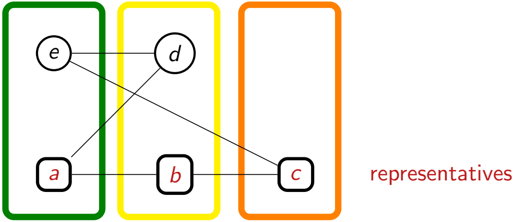
- $\bar{N}(v)$ is the set of vertices not adjacent to v
- number of variables: $|(V \times V) \setminus E|$ and constraints: up to $|V| + |V||E|$
- simple and compact
- problem: symmetry, since any of the vertices can be representative

Campêlo, Corrêa and Frota 2004, Campêlo, Campos and Corrêa 2006

Asymmetric Representatives Formulation

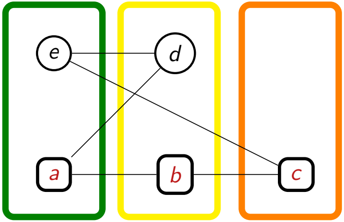
- Let $v_1, v_2, \cdots, v_i, \cdots, v_j, \cdots, v_{|V|}$ be a linear ordering of the vertices



Campêlo, Corrêa and Frota 2004, Campêlo, Campos and Corrêa 2006

Asymmetric Representatives Formulation

- Let $v_1, v_2, \cdots, v_i, \cdots, v_j, \cdots, v_{|V|}$ be a linear ordering of the vertices
- v_i may represent v_j only if $i < j$
- variable for every ordered pair of non-adjacent vertices,



representatives

Campêlo, Corrêa and Frota 2004, Campêlo, Campos and Corrêa 2006

Asymmetric Representatives Formulation

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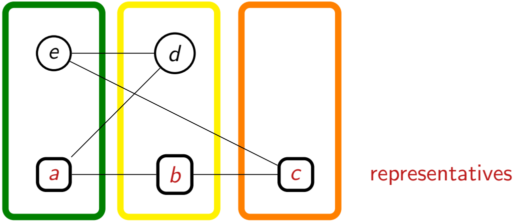
- v_i may represent v_j only if $i < j$
- variable for every ordered pair of non-adjacent vertices,
- $\Rightarrow$ reduces symmetry and number of variables



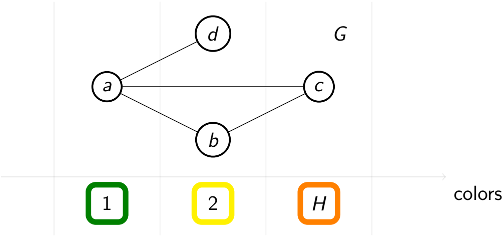
Campêlo, Corrêa and Frota 2004, Campêlo, Campos and Corrêa 2006

Overview

Partial Ordering based Model

Binary variables

- Let $1, 2, \ldots, H$ be a linear ordering of the colors



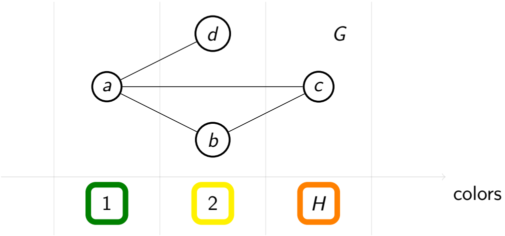
Partial Ordering based Model

Binary variables

- Let $1, 2, \ldots, H$ be a linear ordering of the colors
- For each vertex $v \in V$ and color $i = 1, \ldots, H$ introduce variables

$$l_{v,i} = \begin{cases} 1 & color(v) < i \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad r_{i,v} = \begin{cases} 1 & i < color(v) \\ 0 & \text{otherwise} \end{cases}$$

- Example:
  $l_{a,2} = 1,\ r_{2,a} = 0$
  $l_{b,2} = 0,\ r_{2,b} = 0$



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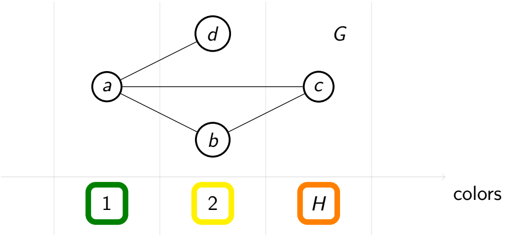
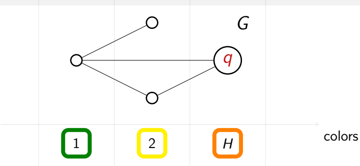
Connection to assignment variables: $x_{v,i} = 1 - (l_{v,i} + r_{i,v})$

- Example:
  $l_{a,2} = 1,\ r_{2,a} = 0$
  $l_{b,2} = 0,\ r_{2,b} = 0$



Partial Ordering based Model



$$\min\ 1 + \sum_{1 \le i \le H} r_{i,q}$$

$$
\begin{aligned}
\text{s.t. } & l_{v,1} = r_{H,v} && = 0 && \forall v \in V && (13)\\
& r_{i-1,v} - r_{i,v} && \ge 0 && \forall v \in V,\ i = 2,\dots,H && (14)\\
& r_{i-1,v} + l_{v,i} && = 1 && \forall v \in V,\ i = 2,\dots,H && (15)\\
& (r_{i,u} + l_{u,i}) + (r_{i,v} + l_{v,i}) && \ge 1 && \forall uv \in E,\ i = 1,\dots,H && (16)\\
& r_{i,q} - r_{i,v} && \ge 0 && \forall v \in V,\ i = 1,\dots,H-1 && (17)\\
& r_{i,v}, l_{v,i} \in \{0,1\} && && \forall v \in V,\ i = 1,\dots,H &&
\end{aligned}
$$

q is an arbitrary vertex chosen from V which we will fix to the largest occupied color class

Jabrayilov and Mutzel 2018

Partial Ordering based Model

- Number of variables: $2H|V|$ and constraints: up to $3H|V| + H|E|$

Jabrayilov and Mutzel 2018, Jabrayilov and Mutzel 2022

Partial Ordering based Model

- Number of variables: $2H|V|$ and constraints: up to $3H|V| + H|E|$
- Equations can be used to reduce the model.
- The reduced model has $(H-1)|V|$ variables only.

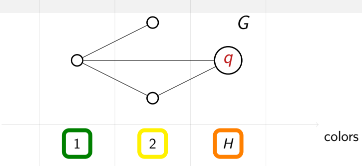
Jabrayilov and Mutzel 2018, Jabrayilov and Mutzel 2022

Partial Ordering based Model

- Number of variables: $2H|V|$ and constraints: up to $3H|V| + H|E|$
- Equations can be used to reduce the model.
- The reduced model has $(H-1)|V|$ variables only.
- Model can be strengthened further $\Rightarrow$ POP2

Jabrayilov and Mutzel 2018, Jabrayilov and Mutzel 2022

Partial Ordering based Model POP2

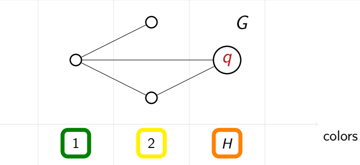


$$
\begin{array}{llll}
\min & 1 + \sum_{1 \le i \le H} r_{i,q} & & \\
\text{s.t.} & r_{H,v} & = 0 & \forall\ v \in V \\
& r_{i-1,v} - r_{i,v} & \ge 0 & \forall\ v \in V,\ i = 2, \dots, H \\
& r_{i,u} + r_{1,v} & \ge 2 - r_{1,q} & \forall\ uv \in E \\
& (r_{i-1,u} - r_{i,u}) + (r_{i-1,v} - r_{i,v}) & \le r_{i-1,q} & \forall\ uv \in E,\ i = 2, \dots, H \\
& r_{i,q} - r_{i,v} & \ge 0 & \forall\ v \in V,\ i = 1, \dots, H-1 \\
& r_{i+1,q} - r_{i,v} & \ge 0 & \forall\ v \in N(q),\ i = 1, \dots, H-1 \\
& r_{i,v}, l_{v,i} \in \{0,1\} & & \forall\ v \in V,\ i = 1, \dots, H
\end{array}
$$

with growing density, the POP models have more non-zero elements than that of ASS $\Rightarrow$ Hybrid model

Hybrid Partial Ordering based Model POPH2

Introducing assignment variables:



$$
\begin{aligned}
\min \quad & 1 + \textstyle\sum_{1 \le i \le H} r_{i,q} & & \\
\text{s.t.} \quad & r_{H,v} = 0 & & \forall v \in V \\
& x_{v,1} = 1 - r_{1,v} & & \forall v \in V \\
& x_{v,i} = r_{i-1,v} - r_{i,v} & & \forall v \in V, i = 2, \ldots, H \\
& x_{u,1} + x_{v,1} \le r_{1,q} & & \forall uv \in E \\
& x_{u,i} + x_{v,i} \le r_{i-1,q} & & \forall uv \in E, i = 2, \ldots, H \\
& r_{i,q} - r_{i,v} \ge 0 & & \forall v \in V,\ i = 1, \ldots, H \\
& r_{i+1,q} - r_{i,v} \ge 0 & & \forall v \in N(q),\ i = 1, \ldots, H-1 \\
& x_{v,i}, r_{i,v} \in \{0, 1\} & & \forall v \in V,\ i = 1, \ldots, H
\end{aligned}
$$

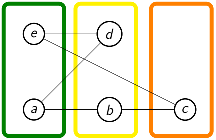
Jabrayilov and Mutzel 2018, Jabrayilov and Mutzel 2022

Overview

Set Covering based Model

Observation

- The vertices of the same color build a stable (independent) set.
- **Idea:** Find a small set of stable sets $S_1, \cdots, S_k$, that cover all vertices, i.e., $S_1 \cup \cdots \cup S_k = V$.



stable sets

Mehrotra and Trick 1996, Malaguti, Monaci and Toth 2011, Held, Cook and Sewell 2011, Held, Cook, and Sewell 2012

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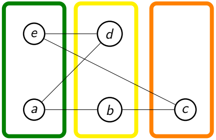
- **Idea:** Find a small set of stable sets $S_1, \cdots, S_k$, that cover all vertices, i.e., $S_1 \cup \cdots \cup S_k = V$.

Binary variables

- Let S be the set of all maximal stable sets. For each $s \in S$ define a variable $x_s = \begin{cases} 1 & \text{stable set } s \text{ is selected} \\ 0 & \text{otherwise.} \end{cases}$



stable sets

Mehrotra and Trick 1996, Malaguti, Monaci and Toth 2011, Held, Cook and Sewell 2011, Held, Cook, and Sewell 2012

Set covering based Model

$$\min \sum_{s \in S} x_s$$

$$s.t. \sum_{s \in S : v \in s} x_s \geq 1 \qquad \forall v \in V$$

$$x_s \in \{0, 1\} \qquad \forall s \in S.$$

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- Number of variables: exponential; number of constraints: $|V|$

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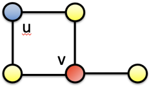
- Number of variables: exponential; number of constraints: $|V|$
- $\Rightarrow$ Branch-and-price algorithms necessary for computation (column generation)
- previous models have polynomial size, and can be computed directly (branch-and-cut)

Mehrotra and Trick 1996, Malaguti, Monaci and Toth 2011, Held, Cook and Sewell 2011, Held, Cook, and Sewell 2012

Overview

Preprocessing

The following techniques are well known:



Preprocessing: domination

- vertex u is dominated by another vertex v, if $N(u) \subseteq N(v)$.
- $\Rightarrow$ u can be deleted
- at the end: u can get the color of v

Preprocessing

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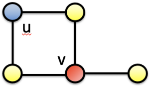

Preprocessing: domination

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Preprocessing: precoloring

- Fix variables in ASS and POP models by precoloring a clique Q with $\max(|Q|H + |\delta(Q)|)$.
- Choose q in POP models from the clique, and color remaining vertices in Q with colors $1, \ldots, |Q| - 1$.

Implementation

Heuristically compute small H

- small upper bound H leads to smaller number of variables in ASS and POP models
- we use heuristics in networkx

Implementation

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Implementation

- Gurobi-python API
- branching priority option: high degree

Implementation by A. Jabrayilov

Computational Study 2018: Simple Methods

Test Set Up

- Gurobi 6.5 single-threadedly on Intel Xeon E5-2640, 2.60GHz with 64 GB of memory, Ubuntu Linux 14.04

Computational Study 2018: Simple Methods

Test Set Up

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- ASS: Assignment model [Mendez & Zabala, 2008]
- POP: Partial Ordering based model [Jabrayilov, Mutzel, 2018]
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Test instances

- hardest instances (68) from the DIMACS benchmark set and the GPIA graphs: $(|V|, |E|) \leq (10\,000, 990\,000)$

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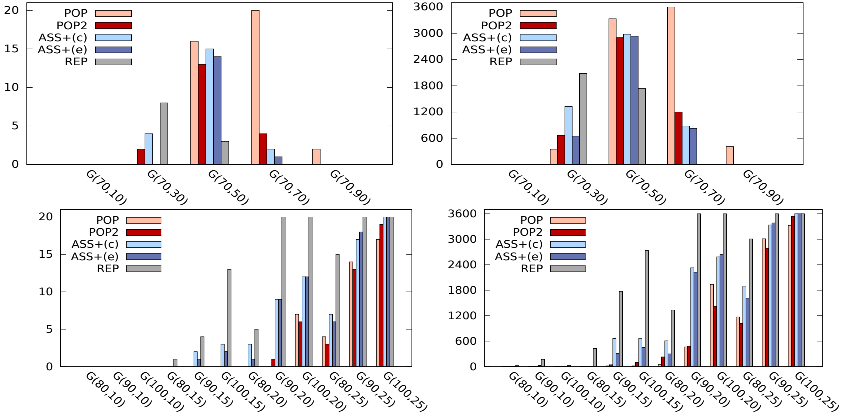
Test instances

- hardest instances (68) from the DIMACS benchmark set and the GPIA graphs: $(|V|, |E|) \leq (10\,000, 990\,000)$
- 340 randomly generated Erdós-Rényi graphs $G(n, p)$ (20 each): $n = |V| = 70, 80, 90, 100$ and edge probability $p \in \{0.1, 0.25\}$

Results for random graphs $G(n,p) \leq G(100, 0.25)$



no. of unsolved instances

avg. time [sec]

Experiments conducted by A. Jabrayilov 2018

Results for 68 DIMACS instances 2018

- COV: Set-Covering model [Malaguti et al. 2011]
- taken from paper, processor was $\approx 1.3\times$ slower

	REP	ASS	POPH	COV
solved	15	21	25	25
avg. time [sec]	170	171	182	1196

Experiments conducted by A. Jabrayilov 2018

Experimental Results 2022:

Implementation and Test Set Up

- Gurobi 6.5 single-threadedly on Intel Xeon E5-2640, 2.60GHz with 64 GB of memory, Ubuntu Linux 18.04
- time limit of 3 hours

Experiments conducted by A. Jabrayilov 2022

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Experiments conducted by A. Jabrayilov 2022

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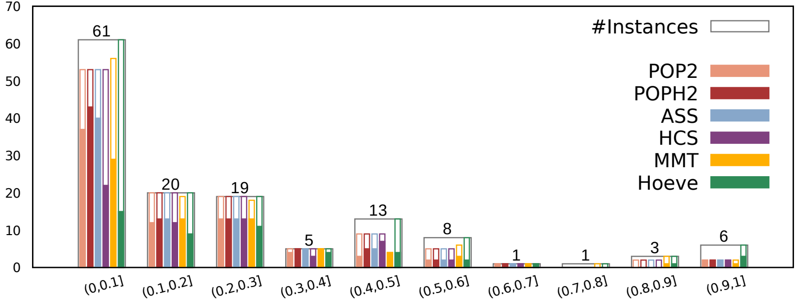
Test instances

- all instances with $|E| \leq 100\,000$ from the DIMACS benchmark set (in total 116 from 137):

Experiments conducted by A. Jabrayilov 2022

Results for 116 DIMACS instances 2022

- HCS: Set covering model, code by Held, Cook, Sewell, 2012
- MMT: Set covering model, from paper by Malaguti, Monaci, Toth, 2011, processor was $\approx 1.3\times$ slower
- Hoeve: OBDD approach, from paper by van Hoeve 2022



no. solved instances depending on graph density $\frac{2|E|}{|V|(|V|-1)}$

Experiments conducted by A. Jabrayilov 2022

Overview

Conclusion from Experiments

For the tested instances and our test setup, we conclude:

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- POPH2 dominates the ASS model.

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For the tested instances and our test setup, we conclude:

- POPH2 dominates the ASS model.
- On the sparse instances with density ≤ 0.1, POPH2 dominates the set covering approaches HCS and MMT.

Conclusion from Experiments

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- The representative models REP and AREP dominate ASS and POP2H for graphs with density larger than 0.5
- POPH2 was the only model that solved all five DIMACS GPIA graphs (estimation of sparse Jacobian matrix problem). POPH2 is the first ILP model solving instance abb313GPIA ($n = 1555$, $|E| = 53356$).

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Source: https://www.inc.com/chris-matyszczyk