Exercise Set 13

Exercise 13.1. Show that for every $k \in \mathbb{N}$ there exists a simple graph G of minimum degree k such that G has exactly one perfect matching.

(3 points)

Exercise 13.2. Let $k \in \mathbb{N}$, $k \geq 1$, and suppose G is a k-regular and $(k-1)$-edge-connected graph with an even number of vertices, and with edge weights $c : E(G) \to \mathbb{R}$. Show that there is a perfect matching M in G with
$c(M) \leq (1/k) \cdot c(E(G))$.

(4 points)

Exercise 13.3. Let G be an undirected graph and and $n \coloneqq |V(G)|$. Prove that the following linear inquality system with $\mathcal{O}(n^3)$ variables and constraints describes a polytope whose orthogonal projection onto the x-variables yields the spanning tree polyptope P_G of G.

$$
\begin{aligned}
x_e &\geq 0 && (e \in E(G)) \\
z_{u,v,w} &\geq 0 && (\{u,v\} \in E(G), w \in V(G)) \\
z_{u,v,w} + z_{v,u,w} &= x_e && (e = \{u,v\} \in E(G), w \in V(G)) \\
\sum_{\{u,v\} \in \delta_G(v)} z_{u,v,w} &= 1 && (v \in V(G), w \in V(G) \setminus \{v\}) \\
\sum_{e \in E(G)} x_e &= n-1
\end{aligned}
$$

(5 points)

Exercise 13.4. Let $G = (s + V, E)$ be a k-edge-connected digraph ($k \geq 1$) with $|\delta^-_G(s)| = |\delta^+_G(s)|$ and $|\delta^+_G(U)|, |\delta^-_G(U)| \geq k$ for all $\emptyset \neq U \subsetneq V$. Then for every edge $e = (s,t) \in E$, there is an edge $(u,s) \in E$ such that the digraph $G' = (s+V, E \setminus \{(s,t),(u,s)\} \cup \{(u,t)\})$ satisfies $|\delta^+_{G'}(U)|, |\delta^-_{G'}(U)| \geq k$ for all $\emptyset \neq U \subsetneq V$. (*Hint: You may want to prove first that $X \cup Y$ has a tight in- or out-degree for any two sets $X, Y \ni t$ with tight in- or out-degree.*)

(4 points)

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Note: This is a "bonus" sheet, i.e. the 16 points which can be obtained here do not increase the number of points required for exam admission. They are intended for groups with only a few missing points as well as for general exam preparation.

Deadline: January 22$^{\text{nd}}$, 12:00 noon. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws23/cows23.html`

In case of any questions feel free to contact me at schuerks@dm.uni-bonn.de.