Professor Dr. Stephan Held
Malte Schürks

Exercise Set 11

Exercise 11.1. Given an undirected graph $G = (V, E)$, we can find an orientation $D = (V, A)$ of G in linear time such that for each $u, v \in V$, for which G has two edge-disjoint u-v-paths, D has a directed u-v-path.

(3 points)

Exercise 11.2. (i) Let $\mathcal{A} = (A_1, \dots, A_n)$ be a family of finite sets. A set T is called traversal of $\mathcal{A}$ if $\exists$ distinct $a_i \in A_i (i \in [n])$ such that $T = \{a_1, \dots, a_n\}$ (thus, $|T| = n$). Prove that $\mathcal{A}$ has a transversal if and only if $|\bigcup_{i \in I} A_i| \geq |I|$ for all $I \subseteq [n]$. (Hint: Marriage theorem)

(ii) Let $G = (V, E)$ be an undirected graph and let $l : V \rightarrow \in \mathbb{Z}_{\geq 0}$. Then G has an orientation $D = (V, A)$ with $\delta_A^-(v) \geq l(v)$ for each $v \in V$ if and only if $|\bigcup_{v \in U} \delta_G(v)| \geq l(U)$ for each $U \subseteq V$

(iii) Let $G = (V, E)$ be an undirected graph and let $u : V \rightarrow \mathbb{Z}_{\geq 0}$. Then G has an orientation $D = (V, A)$ with $\delta_A^-(v) \leq u(v)$ for each $v \in V$ if and only if $|E(G[U])| \leq u(U)$ for each $U \subseteq V$.

(Hint: you may assume (i) to prove (ii), and (ii) to prove (iii))

(1+2+2 points)

Exercise 11.3. Let (V, E) be a forest. Define $f : 2^E \rightarrow \mathbb{Z}$ as

$$f(X) := \sum_{T \in \mathcal{T}(V,X)} \text{diameter}(T) \quad (X \subseteq E),$$

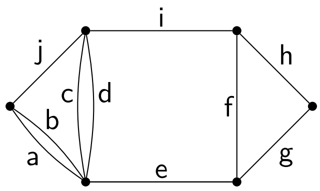
where $\mathcal{T}(V, X)$ denotes the set of connected components, i.e. maximal trees, in (V, X), and diameter(T) is the length of the longest path in T. Prove that f is submodular.

(4 points)

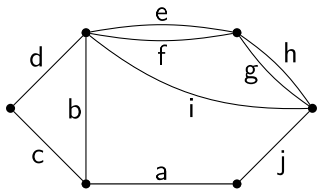
Exercise 11.4. We call a graph *christmas tree* if it has no cycles and its edge set can be partitioned into sets $E_0 \dot{\cup} \dots \dot{\cup} E_k$ such that $(V(E_0), E_0)$ is a path and, for $1 \leq i \leq k$, $(V(E_i), E_i)$ is a path with $|E_i| \leq 2$ and one endpoint in $V(E_0)$. A graph whose connected components are christmas trees is called festive. Prove or disprove:

(i) If $G = (V, E)$ is a graph and $\mathcal{F} := \{F \subseteq E \,|\, (V, F) \text{ is a festive graph}\}$, then $(E, \mathcal{F})$ is a matroid.

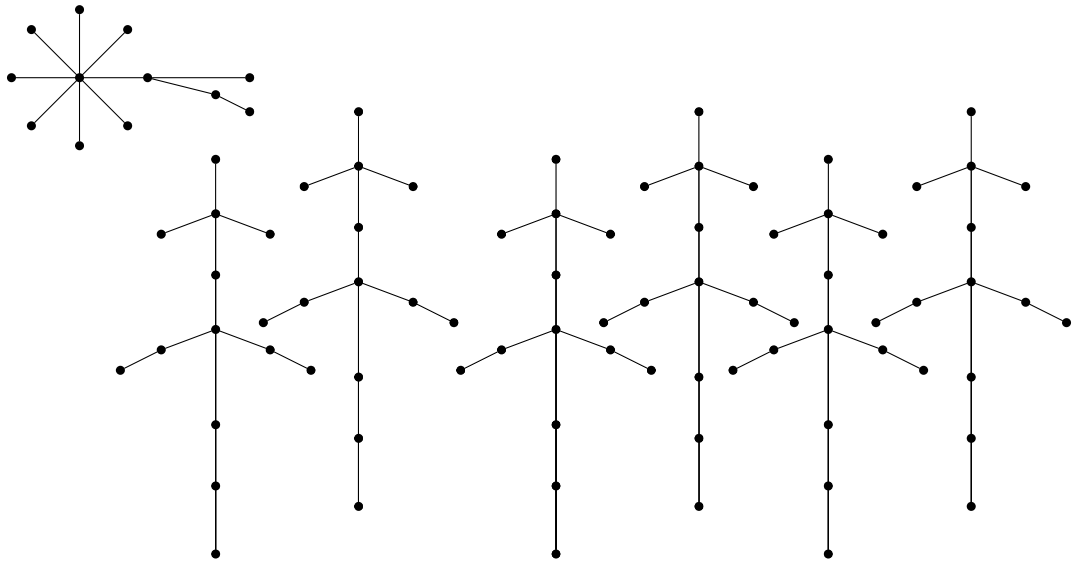
(ii) There exists a set J which is the edge set of a spanning christmas tree in both G_1 and G_2.



G_1



G_2



A festive graph.

(2+2 points)

Deadline: January 11th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws23/cows23.html`

In case of any questions feel free to contact me at schuerks@or.uni-bonn.de.