Exercise Set 10

Exercise 10.1. Prove that the following constructions yield submodular functions f.

(i) For $a \in \mathbb{R}^E$ and $a_0 \in \mathbb{R}$, define $f(S) := a_0 + \sum_{i \in S} a_i$.

(ii) Let $g : 2^E \to \mathbb{R}$ be non-decreasing submodular and $k \in \mathbb{R}$, define $f(S) := \min(g(S), k)$ for all $S \subseteq E$.

(iii) Let $g : 2^E \to \mathbb{R}$ be submodular, define $f(S) := g(E \setminus S)$ for all $S \subseteq E$.

(iv) Let $f_1, f_2 : 2^E \to \mathbb{R}$ be submodular, define $f := f_1 + f_2$.

(1+1+1+1 points)

Exercise 10.2. Let $f, g : 2^E \to \mathbb{R}$ be two submodular functions. Show:

(i) If f and g are also non-decreasing with $f(\emptyset) = g(\emptyset) = 0$, then $P(g + f) = P(f) + P(g) := \{x + y : x \in P(f), y \in P(g)\}$.

(ii) If $f - g$ is non-decreasing, then $\min\{f, g\}$ is submodular.

(2+2 points)

Exercise 10.3. Let $f : 2^E \to \mathbb{Z}$ be submodular, non-decreasing with $f(\emptyset) = 0$, and let $r : 2^E \to \mathbb{Z}$ be defined as

$$r(S) := \min\{f(Q) + |S \setminus Q| \ : \ Q \subseteq S\}$$

for $S \subseteq E$. Prove that r is the rank function of a matroid, i.e. it is submodular, non-decreasing, and $r(S) \leq |S|$.

(4 points)

There is one more exercise on the flip side!

Exercise 10.4. Let $f\colon 2^U \to \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$, and let $B(f)$ denote its base polyhedron. Prove that

$$\begin{aligned}
&\min\{f(X) : X \subseteq U\} \\
&= \max\left\{\sum_{u\in U} z_u : z \in \mathbb{R}^U \text{ with} \sum_{u\in A} z_u \le \min\{0, f(A)\} \text{ for all } A \subseteq U\right\} \\
&= \max\left\{\sum_{u\in U} \min\{0, y_u\} : y \in B(f)\right\}.
\end{aligned}$$

(4 points)

Deadline: December 21th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws23/cows23.html`

In case of any questions feel free to contact me at held@dm.uni-bonn.de.