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Exercise Set 9

Exercise 9.1. Let U be a finite set and $f\colon 2^U \to \mathbb{R}$. Prove that f is submodular if and only if $f(X \cup \{y,z\}) - f(X \cup \{y\}) \leq f(X \cup \{z\}) - f(X)$ for all $X \subseteq U$ and $y, z \in U$ with $y \neq z$.

(4 points)

Exercise 9.2. Let E be a finite set and $f\colon 2^E \to \mathbb{R}$ a function with $f(\emptyset) = 0$ and f' its Lovász extension. Prove Lemma 4.27 and Theorem 4.28 from the lecture, i.e. that

(i) if f is submodular and non-decreasing, then for all $x \in [0,1]^E$

$$f'(x) = \max\{x^T y \colon y \in P(f)\};$$

(ii) f is submodular if and only if f' is convex.

(4+3 points)

Exercise 9.3. Let $f\colon 2^U \to \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$. Let the *base polyhedron* of f be defined as

$$\Big\{x \in \mathbb{R}^U : x(A) \leq f(A) (A \subseteq U), x(U) = f(U)\Big\}$$

Prove that the set of vertices of the base polyhedron of f is precisely the set of vectors $b^{\prec}$ for all total orders $\prec$ of U, where

$$b^{\prec}(u) := f\Big(\{v \in U : v \preceq u\}\Big) - f\Big(\{v \in U : v \prec u\}\Big) \qquad (u \in U).$$

(5 points)

Deadline: December 14th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws23/cows23.html`

In case of any questions feel free to contact me at schuerks@or.uni-bonn.de.