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Exercise Set 7

Exercise 7.1. Let $\lambda_{ij}, 1 \leq i,j \leq n$, be nonnegative numbers with $\lambda_{ij} = \lambda_{ji}$ and $\lambda_{ik} \geq \min\{\lambda_{ij}, \lambda_{jk}\}$ for any three distinct indices $i,j,k \in \{1,\dots,n\}$. Show that there exists a graph G with $V(G) = \{1,\dots,n\}$ and capacities $u\colon E(G) \to \mathbb{R}_+$ such that the local edge-connectivities are precisely the λ_{ij}.

Hint: Consider a maximum weight spanning tree in (K_n, c), where $c(\{i,j\}) := \lambda_{ij}$.

(4 points)

Exercise 7.2. For an undirected graph G, let P_G denote the spanning-tree polytope of G and

$$Q_G := \left\{ x \in [0,1]^{E(G)} : \sum_{e \in E(G)} x_e = |V(G)| - 1 \ , \sum_{e \in \delta(X)} x_e \geq 1 \text{ for } \emptyset \neq X \subsetneq V(G) \right\}.$$

Prove:

(i) $P_G \subseteq Q_G$ for every graph G.

(ii) There exists a graph G with $P_G \neq Q_G$.

(1+1 points)

Exercise 7.3. Let G be a graph, $u\colon E(G) \to \mathbb{N} \cup \{\infty\}$ and $b\colon V(G) \to \mathbb{N}$. Show that (G,u) has a perfect b-matching if and only if for any two disjoint subsets $X, Y \subseteq V(G)$ the number of connected components C in $G - X - Y$ for which $\sum_{c \in V(C)} b(c) + \sum_{e \in E_G(v(C),Y)} u(e)$ is odd does not exceed

$$\sum_{v \in X} b(v) + \sum_{y \in Y} \left(\sum_{e \in \delta(y)} u(e) - b(y) \right) - \sum_{e \in E_G(X,Y)} u(e) \qquad (1)$$

(6 points)

Exercise 7.4. Let G be an undirected graph and $T \subseteq V(G)$ such that $|T|$ is even. Show that the convex hull of incidence vectors of T-joins in G is given by the set P of all $x \in [0,1]^{E(G)}$ satisfying

$$\sum_{e\in F}(1-x_e)+\sum_{e\in\delta(X)\setminus F} x_e \geq 1 \forall X \subseteq V(G),\ F \subseteq \delta(X) \text{ such that } |X\cap T| + |F| \text{ is odd.}$$

Hint: First show that the incidence vector of every T-join in G satisfies the given constraints. Next, show that every vertex of P is the incidence vector of a T-join in G by showing that for each cost function $c \in \mathbb{R}^{E(G)}$, there exists a T-join J such that the incidence vector χ^J of J minimizes $c^T x$ over P. To this end, let $E^- := \{e \in E(G) : c(e) < 0\}$ and, for a vector $x \in P$, consider the vector $\bar{x}$ given by

$$\bar{x}_e = \begin{cases} x_e & , e \in E(G)\setminus E^- \\ 1 - x_e & , e \in E^- \end{cases}$$

and prove that $\bar{x}$ is contained in the up-hull of the $T\Delta\text{odd}(E^-)$-polyhedron of G.

(4 points)

Deadline: November 30th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws23/cows23.html`

In case of any questions feel free to contact me at schuerks@or.uni-bonn.de.