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Exercise Set 6

Exercise 6.1. The UNDIRECTED MINIMUM MEAN-WEIGHT CYCLE PROBLEM is the following: Given an undirected graph G with edge-weights $c : E(G) \to \mathbb{R}$, find a cycle C whose mean-weight $c(E(C))/|E(C)|$ is minimum, or determine that G is acyclic. Consider the following algorithm for the UNDIRECTED MINIMUM MEAN-WEIGHT CYCLE PROBLEM: First determine with a linear search whether G has cycles or not, and if not return with this information.
Let $\gamma := \max\{c(e) : e \in E(G)\}$ and define a new edge-weight function via $c'(e) := c(e) - \gamma$. Let $T := \emptyset$. Now iterate the following: Find a minimum c'-weight T-join J with a polynomial (black-box) algorithm. If $c'(J) = 0$, return any zero-c'-weight cycle. Otherwise, let $\gamma' := c'(J)/|J|$, reset c' via $c'(e) \leftarrow c'(e) - \gamma'$, and continue.

Show that this algorithm works correctly and runs in polynomial time. Also, explain how to get the cycle to be returned in the case $c'(J) = 0$.

(4 points)

Exercise 6.2. Let G be an undirected graph and $T \subseteq V(G)$ with $|T| = 2k$ even. Prove that the minimum cardinality of a T-cut in G equals the maximum of $\min_{i=1}^{k} \lambda_{s_i,t_i}$ over all pairings $T = \{s_1, t_1, \ldots, s_k, t_k\}$, where $\lambda_{s,t}$ denotes the maximum number of pairwise edge-disjoint s-t-paths.

(4 points)

Exercise 6.3. Given an undirected graph G and disjoint sets $S_e, S_o \subseteq V(G)$, a *partial* (S_e, S_o)*-join* is a set $J \subseteq E(G)$ such that $|\delta(v) \cap J|$ is even for every $v \in S_e$ and odd for every $v \in S_o$. (In particular, a T-join is the same as a partial $(V(G) \setminus T, T)$-join.) Consider the MINIMUM WEIGHT PARTIAL (S_e, S_o)-JOIN PROBLEM: Given an undirected graph G with edge-weights $c : E(G) \to \mathbb{R}_{\geq 0}$ and disjoint sets $S_e, S_o \subseteq V(G)$, find a partial (S_e, S_o)-join of minimum weight, or determine that none exists. Show that this problem can be linearly reduced to the MINIMUM WEIGHT T-JOIN PROBLEM.

(4 points)

Exercise 6.4. Let G be a connected graph, $c : E(G) \to \mathbb{R}$ and $T \subseteq V(G)$. Note that $|T|$ may be either even or odd. Consider the problem of finding a T^*-join J^* with $|T^* \Delta T| \leq 2$ such that $c(J^*)$ is minimum among all such J^*. Show that this problem can be linearly reduced to the MINIMUM WEIGHT T-JOIN PROBLEM.

Remark: This problem arises when adapting Christofides' algorithm to the problem of finding a shortest Hamiltonian path with up to one endpoint of the path given in the input.

(4 points)

Deadline: November 23^{rd}, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws23/cows23.html`

In case of any questions feel free to contact me at schuerks@or.uni-bonn.de.