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Exercise Set 5

Exercise 5.1. Let $G = (V, E)$ be an undirected graph and Q its fractional perfect matching polytope, which is defined by

$$Q = \{x \in \mathbb{R}^E : x_e \geq 0 \ (e \in E), \sum_{e \in \delta(v)} x_e = 1 \ (v \in V)\}.$$

Prove that a vector $x \in Q$ is a vertex of Q if and only if there exist vertex disjoint odd circuits $C_1, \ldots, C_k$ and a perfect matching M in $G - (V(C_1) \cup \ldots \cup V(C_k))$ such that

$$x_e = \begin{cases} \frac{1}{2} & \text{if } e \in E(C_1) \cup \ldots \cup E(C_k), \\ 1 & \text{if } e \in M, \\ 0 & \text{otherwise.} \end{cases}$$

(4 points)

Exercise 5.2. Consider the Minimum Cost Edge Cover Problem: Given a graph G with weights $c : E(G) \to \mathbb{R}_{\geq 0}$, find an edge cover $F \subseteq E(G)$ that minimizes $\sum_{e \in F} c(e)$. Show that the Minimum Cost Edge Cover Problem can be linearly reduced to the Minimum Weight Perfect Matching Problem.
(4 points)

Exercise 5.3. Let G be a graph and $T \subseteq V(G)$ with $|T|$ even. Prove:

(i) A set $F \subseteq E(G)$ intersects every T-join if and only if it contains a T-cut.

(ii) A set $F \subseteq E(G)$ intersects every T-cut if and only if it contains a T-join.

(4 points)

Exercise 5.4. Let G be an undirected graph and $c_1, c_2 \colon E(G) \to \mathbb{R}$ be two weight functions. Let $\mathcal{M}$ be the set of all matchings that have maximum weight with respect to c_1. How can we find, in polynomial time, a matching $M \in \mathcal{M}$ such that $c_2(M)$ is maximum among all matchings in $\mathcal{M}$? Can you devise a strongly polynomial algorithm? (For this, in particular, the algorithm should work for

arbitrary real numbers, assuming that we can perform addition, subtraction and comparison.)

Note: You can use the fact that there exists a strongly polynomial algorithm for the Maximum Weight Matching Problem.

(4 points)

Deadline: November 16th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws23/cows23.html`

In case of any questions feel free to contact me at schuerks@or.uni-bonn.de.