Exercise Set 3

Exercise 3.1. Let G be a k-vertex-connected graph which has neither a perfect nor a near-perfect matching.

(i) Show that $\nu(G) \geq k$.

(ii) Show that $\tau(G) \leq 2 \cdot \nu(G) - k$.

(2+2 points)

Exercise 3.2. Let G be a graph and M a matching in G that is not maximum. In this exercise we use the terminology *disjoint subgraphs/paths/circuits* and mean it quite literally: Two subgraphs are *disjoint* if they have no edges and no vertices in common. (Note that the term *vertex-disjoint paths* is often used to mean that two paths have no *inner*-vertices in common, but possibly endpoints.)

(i) Show that there are $\nu(G) - |M|$ disjoint M-augmenting paths in G.

(ii) Show the existence of an M-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

(iii) Let P be a shortest M-augmenting path in G and P' an $(M \Delta E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + 2 \cdot |E(P) \cap E(P')|$.

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then P_i and P_j are disjoint.

(v) Show that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains less than $2\sqrt{\nu(G)} + 1$ different numbers.

From now on, let G be bipartite and set $n := |V(G)|$ and $m := |E(G)|$.

(vi) Given a non-maximum matching M in G show that we can find in $O(n+m)$-time a family $\mathcal{P}$ of disjoint shortest M-augmenting paths such that if M' is the matching obtained by augmenting M over every path in $\mathcal{P}$, then

$$\texttt{min}\{|E(P)| : P \text{ is an } M'\text{-augmenting path}\} > \texttt{min}\{|E(P)| : P \text{ is an } M\text{-augmenting path}\}$$

(vii) Describe an algorithm with runtime $O(\sqrt{n}(m+n))$ that solves the Cardinality Matching Problem in bipartite graphs.

(1+1+2+2+2+3+1=12 points)

Deadline: November 2nd, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws23/cows23.html`

In case of any questions feel free to contact me at schuerks@or.uni-bonn.de.