Exercise Set 1

Exercise 1.1. Find an infinite counterexample to Hall's Theorem. More precisely: Find a bipartite graph $G = (A \dot{\cup} B, E)$ with $A \cong \mathbb{N}$, $B \cong \mathbb{N}$, and $|N(S)| \geq |S|$ for every $S \subseteq A$ and every $S \subseteq B$ such that G does not contain a perfect matching.

(4 points)

Exercise 1.2. Let G be a bipartite graph.

(a) Let $V(G) = A \dot{\cup} B$ be a bipartition of G.
If $A' \subseteq A$ and $B' \subseteq B$, and there are a matching $M_{A'}$ covering A' and a matching $M_{B'}$ covering B', show that there must be a matching that covers $A' \cup B'$.

(b) Suppose that for every non-empty $E' \subseteq E(G)$ we have $\tau(G - E') < \tau(G)$. Show that $E(G)$ is a matching in G.

(3+1 points)

Exercise 1.3. Let G be a k-regular bipartite graph.

(a) Prove that G contains k disjoint perfect matchings.
Hint: Use König's Theorem.

(b) Deduce from (a) that the edge set of any bipartite graph of maximum degree k can be partitioned into k matchings.

(2 + 2 points)

Exercise 1.4. Let $\alpha(G)$ denote the size of a maximum stable set in G, and $\zeta(G)$ the minimum cardinality of an edge cover. Prove:

(a) $\alpha(G) + \tau(G) = |V(G)|$ for any graph G,

(b) $\nu(G) + \zeta(G) = |V(G)|$ for any graph G with no isolated vertices,

(c) $\zeta(G) = \alpha(G)$ for any bipartite graph G with no isolated vertices.

(4 points)

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Deadline: October 19$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws23/cows23.html`

In case of any questions feel free to contact me at schuerks@or.uni-bonn.de.