Professor Dr. Stephan Held
Stefan Rabenstein

Exercise Set 12

Exercise 12.1. Let U be a finite set and $f\colon 2^U \to \mathbb{R}$. Prove that f is submodular if and only if $f(X \cup \{y,z\}) - f(X \cup \{y\}) \leq f(X \cup \{z\}) - f(X)$ for all $X \subseteq U$ and $y, z \in U$ with $y \neq z$.

(5 points)

Exercise 12.2. Let $0 < \epsilon < \frac{1}{2}$ be fixed and $n \in \mathbb{N}$ even with $\epsilon n \in \mathbb{N}$. Let $U = \{1, \ldots, n\}$. For any $C \subset U$ with $2|C| = |U|$ consider the functions $g, f_C \colon 2^U \to \mathbb{Z}_+$ defined as follows: For $S \subseteq U$ let $k := |S \cap C|$ and $l := |S \setminus C|$, and let $g(S) := |S||U \setminus S|$ and $f_C(S) := g(S)$ if $|k-l| \leq \epsilon n$ and $f_C(S) := n|S| - 4kl + \epsilon^2 n^2 - 2\epsilon n|k-l|$ if $|k-l| \geq \epsilon n$.

(i) Show that the two definitions of $f_C(S)$ coincide if $|k-l| = \epsilon n$.

(ii) Show that g and f_C are submodular. *Hint:* Use Exercise 12.1.

(iii) Observe that an algorithm is likely to need exponentially many oracle calls to find out which of these functions (g or f_C for some C) is the input.

(iv) Show that the maximum values of g and any f_C differ by a factor more than $2(1-2\epsilon)$.

(3 + 3 + Bonus* + 4 points)

* Bonus points given for (iii) make up for points missing in (i), (ii) and (iv).

Exercise 12.3. Let S be a finite set and let $b_1, b_2 : 2^S \mapsto \mathbb{R}$ be two submodular functions. Furthermore, let S' and S'' be two disjooint copies of S. Set $V = S' \dot\cup S''$ and

$$\mathcal{C} = \{U' : U \subseteq S\} \cup \{S' \cup U'' : U \subseteq S\},$$

where U' and U'' denote the two copies of $U \subseteq S$ in S' and S'', and define $b : \mathcal{C} \mapsto \mathbb{R}_{\geq 0}$ by

$$\begin{aligned} b(U') &:= b_1(U) && \text{for } U \subsetneq S, \\ b(V \setminus U'') &:= b_2(U) && \text{for } U \subsetneq S, \\ b(S') &:= \min\{b_1(S), b_2(S)\}. \end{aligned} \tag{1}$$

(i) Show that $\mathcal{C}$ is a crossing family.

(ii) Show that b is crossing submodular on $\mathcal{C}$.

(2+3 points)

Deadline: January 16, before the lecture.