Professor Dr. Stephan Held
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Exercise Set 1

Exercise 1.1. Let G be a graph and M_1 and M_2 be two inclusion-wise maximal matchings in G. Prove that $|M_1| \leq 2|M_2|$.

(4 points)

Exercise 1.2. Find an infinite counterexample to Hall's Theorem. More precisely: Find a bipartite graph $G = (A \dot{\cup} B, E)$ with $A \cong \mathbb{N}$, $B \cong \mathbb{N}$, and $|\Gamma(S)| \geq |S|$ for every $S \subseteq A$ and every $S \subseteq B$ such that G does not contain a perfect matching.

(4 points)

Exercise 1.3. Let $\alpha(G)$ denote the size of a maximum stable set in G, and $\zeta(G)$ the minimum cardinality of an edge cover. Prove:

(a) $\alpha(G) + \tau(G) = |V(G)|$ for any graph G,

(b) $\nu(G) + \zeta(G) = |V(G)|$ for any graph G with no isolated vertices,

(c) $\zeta(G) = \alpha(G)$ for any bipartite graph G with no isolated vertices.

(1 + 2 + 1 points)

Exercise 1.4. Let G be a bipartite graph with bipartition $V(G) = A \dot{\cup} B$, $A = \{a_1, \dots, a_k\}$, $B = \{b_1, \dots, b_k\}$. For any vector $x = (x_e)_{e \in E(G)}$ we define the matrix $M_G(x) = (m_{ij}^x)_{1 \leq i,j \leq k}$ by

$$m_{ij}^x := \begin{cases} x_e & \text{if } e = \{a_i, b_j\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Its determinant $\det M_G(x)$ is a polynomial in $x = (x_e)_{e \in E(G)}$. Prove that G has a perfect matching if and only if $\det M_G(x)$ is not identically zero.

(4 points)

Deadline: October 17th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws19/co_exercises/exercises.html`

In case of any questions feel free to contact me at rabenstein@or.uni-bonn.de.