# Exercise Set 7

**Exercise 7.1.** Let $G$ be a graph and $T \subseteq V(G)$ with $|T|$ even. Prove:

(i) A set $F \subseteq E(G)$ intersects every $T$-join if and only if it contains a $T$-cut.

(ii) A set $F \subseteq E(G)$ intersects every $T$-cut if and only if it contains a $T$-join.

(3 points)

**Exercise 7.2.** Let $G$ be an undirected graph, $T \subseteq V(G)$ and let $J \subseteq E(G)$ be a $T$-join of minimum cardinality. Denote by $\nu(G,T)$ the maximum cardinality of a family of pairwise disjoint $T$-cuts and by $\tau(G,T)$ the minimum cardinality of a $T$-join. Consider the properties:

(i) $\nu(G,T) = \tau(G,T)$.

(ii) There exist $|J|$ pairwise disjoint cuts $\delta(X_1), \ldots, \delta(X_{|J|})$ with $|\delta(X_i) \cap J| = 1$ for every $i$.

Show that (i) and (ii) are equivalent.

(2 points)

**Exercise 7.3.** Consider the EDGE-DISJOINT PATHS PROBLEM: Given two graphs $G = (V,E)$ and $H = (V,F)$, decide if there exists a family $(P_f)_{f \in F}$ of edge disjoint paths in $G$, where $P_{\{s,t\}}$ is an $s$-$t$-path. This problem is $\mathcal{NP}$-complete even if $(V, E \,\dot\cup\, F)$ is planar.

Use this fact to show that it is $\mathcal{NP}$-complete to decide, given some planar graph $G$ and some $T \subseteq V(G)$, whether $\nu(G,T) = \tau(G,T)$ holds.

(4 points)

**Exercise 7.4.** The Undirected Minimum Mean-Weight Cycle Problem is the following: Given an undirected graph $G$ with edge-weights $c : E(G) \to \mathbb{R}$, find a cycle $C$ whose mean-weight $c(E(C))/|E(C)|$ is minimum, or determine that $G$ is acyclic. Consider the following algorithm for the Undirected Minimum Mean-Weight Cycle Problem: First determine with a linear search whether $G$ has cycles or not, and if not return with this information. Let $\gamma := \max\{c(e) : e \in E(G)\}$ and define a new edge-weight function via $c'(e) := c(e) - \gamma$. Let $T := \emptyset$. Now iterate the following: Find a minimum $c'$-weight $T$-join $J$ with a polynomial (black-box) algorithm. If $c'(J) = 0$, return any zero-$c'$-weight cycle. Otherwise, let $\gamma' := c'(J)/|J|$, reset $c'$ via $c'(e) \leftarrow c'(e) - \gamma'$, and continue.

Show that this algorithm works correctly and runs in polynomial time. Also, explain how to the get the cycle to be returned in the case $c'(J) = 0$.

(4 points)

**Exercise 7.5.** For an undirected graph $G$, let $P_G$ denote the spanning-tree polytope of $G$ and

$$Q_G := \left\{ x \in [0,1]^{E(G)} : \sum_{e \in E(G)} x_e = |V(G)| - 1 \ , \sum_{e \in \delta(X)} x_e \geq 1 \text{ for } \emptyset \neq X \subsetneq V(G) \right\}.$$

Prove:

(i) $P_G \subseteq Q_G$ for every graph $G$.

(ii) There exists a graph $G$ with $P_G \neq Q_G$.

(1+2 points)

**Deadline:** November 30th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws17/co_exercises/exercises.html`

In case of any questions feel free to contact me at silvanus@or.uni-bonn.de.