Linear and Integer Programming

- Time: Tuesdays and Thursdays, 12:15 - 13:55 (with 10 minutes break)
- Place: Gerhard-Konow-Hörsaal, Lennéstr. 2
- Website:
  `www.or.uni-bonn.de/lectures/ws16/lgo_ws16.html`
- Lecture notes and all slides can be found on the website.

Final Examination

- Oral examination
- Dates by appointment.

Exercise Classes

- Exercise classes are two hours per week.
- Assignments are released every Tuesday (starting in the second week).
- There will be programming exercises.
- 50 % of all points in the assignments are required to participate in the exam.
- Students can work in groups of two.
- All participants of a group have to be able to explain their solutions.
- Exercise classes begin in the second week.

Possible Time Slots for the Exercise Classes

1. Mo 10 - 12
2. Tu 14 - 16
3. We 10 - 12
4. We 12 - 14
5. Th 10 - 12
6. Th 14 - 16
7. Th 16 - 18
8. Fr 10 - 12
9. Fr 12 - 14

We will choose two of these time slots.

Application for the exercise classes: See the website

`www.or.uni-bonn.de/lectures/ws16/lgo_uebung_ws16.html`

Modelling Optimization Problems as LPs

Defintion

Let G be a directed graph with capacities $u : E(G) \to \mathbb{R}_{>0}$ and let s and t be two vertices of G. A feasible ***s*-*t*-flow** in (G, u) is a mapping $f : E(G) \to \mathbb{R}_{\geq 0}$ with

- $f(e) \leq u(e)$ for all $e \in E(G)$ and
- $\sum_{e \in \delta^+_G(v)} f(e) - \sum_{e \in \delta^-_G(v)} f(e) = 0$ for all $v \in V(G) \setminus \{s, t\}$.

The **value** of an s-t-flow f is $\text{val}(f) = \sum_{e \in \delta^+_G(s)} f(e) - \sum_{e \in \delta^-_G(s)} f(e)$.

Modelling Optimization Problems as LPs

MAXIMUM-FLOW PROBLEM

Instance: A directed Graph G, capacities $u : E(G) \to \mathbb{R}_{>0}$, vertices $s, t \in V(G)$ with $s \neq t$.

Task: Find an s-t-flow $f : E(G) \to \mathbb{R}_{\geq 0}$ of maximum value.

LP-formulation:

$$
\begin{array}{lrcll}
\max & \displaystyle\sum_{e \in \delta^+_G(s)} x_e - \sum_{e \in \delta^-_G(s)} x_e & & & \\
\text{s.t.} & x_e & \geq & 0 & \text{for } e \in E(G) \\
 & x_e & \leq & u(e) & \text{for } e \in E(G) \\
 & \displaystyle\sum_{e \in \delta^+_G(v)} x_e - \sum_{e \in \delta^-_G(v)} x_e & = & 0 & \text{for } v \in V(G) \setminus \{s, t\}
\end{array}
$$

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
 & \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
 & & 8x_1 & + & 12x_2 & \leq & 7 \\
 & & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Goal: Find an upper bound on the optimum solution value.

Combine constraint 1 and 2:

$$12x_1 + 10x_2 = 2 \cdot (4x_1 + 2x_2) + \frac{1}{2}(8x_1 + 12x_2) \leq 2 \cdot 5 + \frac{1}{2} \cdot 7 = 13.5.$$

Combine constraint 2 and 3:

$$12x_1 + 10x_2 = \frac{7}{6} \cdot (8x_1 + 12x_2) + \frac{4}{3} \cdot (2x_1 - 3x_2) \leq \frac{7}{6} \cdot 7 + \frac{4}{3} \cdot 1 = 9.5.$$

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
& \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
& & 8x_1 & + & 12x_2 & \leq & 7 \\
& & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

General approach: Find numbers $u_1, u_2, u_3 \in \mathbb{R}_{\geq 0}$ such that

$$12x_1 + 10x_2 = u_1 \cdot (4x_1 + 2x_2) + u_2 \cdot (8x_1 + 12x_2) + u_3 \cdot (2x_1 - 3x_2).$$

$\Rightarrow$ $5u_1 + 7u_2 + u_3$ is an upper bound on the value of any solution of (P).

$\Rightarrow$ Chose u_1, u_2, u_3 such that $5u_1 + 7u_2 + u_3$ is minimized.

Duality: Example

$$
\begin{array}{llrcrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
& \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
& & 8x_1 & + & 12x_2 & \leq & 7 \\
& & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Formulation as a linear program:

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & u_3 & & \\
& \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
& & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
& & u_1 & & & & & \geq & 0 \\
& & & & u_2 & & & \geq & 0 \\
& & & & & & u_3 & \geq & 0
\end{array}
$$

Any solution of (D) gives an upper bound for (P).

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
 & \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
 & & 8x_1 & + & 12x_2 & \leq & 7 \\
 & & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Formulation as a linear program:

$$
\begin{array}{llrcrcrcl}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & u_3 & & \\
 & \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
 & & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
 & & u_1 & & & & & \geq & 0 \\
 & & & & u_2 & & & \geq & 0 \\
 & & & & & & u_3 & \geq & 0
\end{array}
$$

Any solution of (D) gives an upper bound for (P).

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
& \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
& & 8x_1 & + & 12x_2 & \leq & 7 \\
& & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Formulation as a linear program:

$$
\begin{array}{llrcrcrcl}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & 1u_3 & & \\
& \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
& & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
& & u_1 & & & & & \geq & 0 \\
& & & & u_2 & & & \geq & 0 \\
& & & & & & u_3 & \geq & 0
\end{array}
$$

Any solution of (D) gives an upper bound for (P).

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
& \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
& & 8x_1 & + & 12x_2 & \leq & 7 \\
& & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Formulation as a linear program:

$$
\begin{array}{llrcrcrcl}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & u_3 & & \\
& \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
& & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
& & u_1 & & & & & \geq & 0 \\
& & & & u_2 & & & \geq & 0 \\
& & & & & & u_3 & \geq & 0
\end{array}
$$

Any solution of (D) gives an upper bound for (P).

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
& \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
& & 8x_1 & + & 12x_2 & \leq & 7 \\
& & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Formulation as a linear program:

$$
\begin{array}{llrcrcrcl}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & u_3 & & \\
& \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
& & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
& & u_1 & & & & & \geq & 0 \\
& & & & u_2 & & & \geq & 0 \\
& & & & & & u_3 & \geq & 0
\end{array}
$$

Any solution of (D) gives an upper bound for (P).

Fourier-Motzkin Elimination I

Given a system of inequalities, check if a solution exists.

$$
\begin{array}{rcrcrcr}
3x & + & 2y & + & 4z & \leq & 10 \\
3x & & & + & 2z & \leq & 9 \\
2x & - & y & & & \leq & 5 \\
-x & + & 2y & - & z & \leq & 3 \\
-2x & & & & & \leq & 4 \\
& & 2y & + & 2z & \leq & 7
\end{array}
$$

First step: Get rid of variable x.

Fourier-Motzkin Elimination II

$$
\begin{array}{rcrcrcr}
3x & + & 2y & + & 4z & \leq & 10 \\
3x & & & + & 2z & \leq & 9 \\
2x & - & y & & & \leq & 5 \\
-x & + & 2y & - & z & \leq & 3 \\
-2x & & & & & \leq & 4 \\
 & & 2y & + & 2z & \leq & 7
\end{array}
$$

is equivalent to

$$
\begin{array}{rcrcrcr}
x & \leq & \frac{10}{3} & - & \frac{2}{3}y & - & \frac{4}{3}z \\
x & \leq & 3 & & & - & \frac{2}{3}z \\
x & \leq & \frac{5}{2} & + & \frac{1}{2}y & & \\
x & \geq & -3 & + & 2y & - & z \\
x & \geq & -2 & & & & \\
 & & 2y & + & 2z & \leq & 7
\end{array}
$$

Fourier-Motzkin Elimination III

$$
\begin{array}{rcrcrcr}
x & \leq & \frac{10}{3} & - & \frac{2}{3}y & - & \frac{4}{3}z \\
x & \leq & 3 & & & - & \frac{2}{3}z \\
x & \leq & \frac{5}{2} & + & \frac{1}{2}y & & \\
x & \geq & -3 & + & 2y & - & z \\
x & \geq & -2 & & & & \\
& & 2y & + & 2z & \leq & 7
\end{array}
$$

This system is feasible if and only if the following system has a solution:

$$
\begin{array}{rcl}
\min\left\{\frac{10}{3} - \frac{2}{3}y - \frac{4}{3}z, \quad 3 - \frac{2}{3}z, \quad \frac{5}{2} + \frac{1}{2}y\right\} & \geq & \max\left\{-3 + 2y - z, \quad -2\right\} \\
2y + 2z & \leq & 7
\end{array}
$$

Fourier-Motzkin Elimination IV

$$
\begin{array}{rcl}
\min\left\{\frac{10}{3}-\frac{2}{3}y-\frac{4}{3}z, \quad 3-\frac{2}{3}z, \quad \frac{5}{2}+\frac{1}{2}y\right\} & \geq & \max\left\{-3+2y-z, \quad -2\right\} \\
2y+2z & \leq & 7
\end{array}
$$

This system can be rewritten in the following way:

$$
\begin{array}{rcrcrcrcrcrcr}
\frac{10}{3} & - & \frac{2}{3}y & - & \frac{4}{3}z & \geq & -3 & + & 2y & - & z \\
\frac{10}{3} & - & \frac{2}{3}y & - & \frac{4}{3}z & \geq & -2 \\
3 & & & - & \frac{2}{3}z & \geq & -3 & + & 2y & - & z \\
3 & & & - & \frac{2}{3}z & \geq & -2 \\
\frac{5}{2} & + & \frac{1}{2}y & & & \geq & -3 & + & 2y & - & z \\
\frac{5}{2} & + & \frac{1}{2}y & & & \geq & -2 \\
& & 2y & + & 2z & \leq & 7
\end{array}
$$

Fourier-Motzkin Elimination V

Conversion in standard form:

$$\begin{array}{rcrcr}
\frac{8}{3}y & + & \frac{1}{3}z & \leq & \frac{19}{3} \\
\frac{2}{3}y & + & \frac{4}{3}z & \leq & \frac{16}{3} \\
\frac{8}{3}y & - & z & \leq & 6 \\
 & & \frac{2}{3}z & \leq & 5 \\
\frac{3}{2}y & - & z & \leq & \frac{11}{2} \\
-\frac{1}{2}y & & & \leq & \frac{9}{2} \\
2y & + & 2z & \leq & 7
\end{array}$$

Iterate these steps and remove *all* variables.

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n. \text{Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \quad : \quad \begin{array}{ccccccc} Ax & + & By & + & Cz & \leq & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \geq & c \\ x & & & & & \geq & 0 \\ & & & & z & \leq & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \quad : \quad \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \geq & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \leq & f \\ u & & & & & \geq & 0 \\ & & & & w & \leq & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n. \text{Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \quad : \quad \begin{array}{ccccccc} Ax & + & By & + & Cz & \leq & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \geq & c \\ x & & & & & \geq & 0 \\ & & & & z & \leq & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \quad : \quad \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \geq & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \leq & f \\ u & & & & & \geq & 0 \\ & & & & w & \leq & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n. \text{Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \; : \; \begin{array}{ccccccc} Ax & + & By & + & Cz & \leq & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \geq & c \\ x & & & & & \geq & 0 \\ & & & & z & \leq & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \; : \; \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \geq & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \leq & f \\ u & & & & & \geq & 0 \\ & & & & w & \leq & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n. \text{Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \quad : \begin{array}{ccccccc} Ax & + & By & + & Cz & \leq & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \geq & c \\ x & & & & & \geq & 0 \\ & & & & z & \leq & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \quad : \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \geq & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \leq & f \\ u & & & & & \geq & 0 \\ & & & & w & \leq & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n.\text{Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \quad : \quad \begin{array}{ccccccc} Ax & + & By & + & Cz & \le & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \ge & c \\ x & & & & & \ge & 0 \\ & & & & z & \le & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \quad : \quad \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \ge & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \le & f \\ u & & & & & \ge & 0 \\ & & & & w & \le & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Max-Flow Problem

G Digraph, $u : E(G) \to \mathbb{R}_{>0}$, $s, t \in V(G)$ with $s \neq t$.

LP-formulation:

$$
\begin{array}{llcll}
\max & \sum_{e \in \delta^+_G(s)} x_e - \sum_{e \in \delta^-_G(s)} x_e & & & \\
\text{s.t.} & x_e & \geq & 0 & \text{for } e \in E(G) \\
 & x_e & \leq & u(e) & \text{for } e \in E(G) \\
 & \sum_{e \in \delta^+_G(v)} x_e - \sum_{e \in \delta^-_G(v)} x_e & = & 0 & \text{for } v \in V(G) \setminus \{s, t\}
\end{array}
$$

Dual LP:

$$
\begin{array}{llcll}
\min & \sum_{e \in E(G)} u(e) y_e & & & \\
\text{s.t.} & y_e & \geq & 0 & \text{for } e \in E(G) \\
 & y_e + z_v - z_w & \geq & 0 & \text{for } e = (v, w) \in E(G) \\
 & z_s & = & -1 & \\
 & z_t & = & 0 &
\end{array}
$$

Max-Flow Problem

G Digraph, $u : E(G) \to \mathbb{R}_{>0}$, $s, t \in V(G)$ with $s \neq t$.

LP-formulation:

$$
\begin{array}{lrcll}
\max & \displaystyle\sum_{e \in \delta^+_G(s)} x_e - \sum_{e \in \delta^-_G(s)} x_e & & & \\
\text{s.t.} & x_e & \geq & 0 & \text{for } e \in E(G) \\
 & x_e & \leq & u(e) & \text{for } e \in E(G) \\
 & \displaystyle\sum_{e \in \delta^+_G(v)} x_e - \sum_{e \in \delta^-_G(v)} x_e & = & 0 & \text{for } v \in V(G) \setminus \{s, t\}
\end{array}
$$

Dual LP:

$$
\begin{array}{lrcll}
\min & \displaystyle\sum_{e \in E(G)} u(e) y_e & & & \\
\text{s.t.} & y_e & \geq & 0 & \text{for } e \in E(G) \\
 & y_e + z_v - z_w & \geq & 0 & \text{for } e = (v, w) \in E(G) \\
 & z_s & = & -1 & \\
 & z_t & = & 0 &
\end{array}
$$

Theorem

Let $P \subseteq \{x \in \mathbb{R}^n \mid Ax = b\}$ be a non-empty polyhedron of dimension $n - \text{rank}(A)$. Let $A'x \leq b'$ be a minimal system of inequalities such that $P = \{x \in \mathbb{R}^n \mid Ax = b, A'x \leq b'\}$. Then, every inequality in $A'x \leq b'$ is facet-defining for P and every facet of P is given by an inequality of $A'x \leq b'$.

Simplex Algorithm: Example I

$$
\begin{array}{rrcrcrcrcrcl}
\max & x_1 & + & x_2 & & & & & & & & \\
s.t. & -x_1 & + & x_2 & + & x_3 & & & & & = & 1 \\
 & x_1 & & & & & + & x_4 & & & = & 3 \\
 & & & x_2 & & & & & + & x_5 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

Initial basis: $\{3,4,5\}$. $\Rightarrow$ $A_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Simplex tableau:

$$
\begin{array}{rclcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

Recent solution: $(0,0,1,3,2)$

Simplex Algorithm: Example I

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

Increase exactly one of the non-basic variables with positive coefficient in the objective function.
We choose x_2. How much can we increase it?

Constraints:
$x_3 = 1 + x_1 - x_2$: x_2 cannot get larger than 1.
$x_4 = 3 - x_1$: no constraint on x_2.
$x_5 = 2 - x_2$: x_2 cannot get larger than 2.

Strictest constraint: $x_3 = 1 + x_1 - x_2$
$\Rightarrow$ Replace 3 by 2 in B.

Simplex Algorithm: Example I

First tableau:

$$\begin{array}{rcrcrcr} x_3 & = & 1 & + & x_1 & - & x_2 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 2 & & & - & x_2 \\ \hline z & = & & & x_1 & + & x_2 \end{array}$$

Replace 3 by 2 in the basis B: $B = \{2, 4, 5\}$:

$x_2 = 1 + x_1 - x_3$.

Second tableau:

$$\begin{array}{rcrcrcr} x_2 & = & 1 & + & x_1 & - & x_3 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 1 & - & x_1 & + & x_3 \\ \hline z & = & 1 & + & 2x_1 & - & x_3 \end{array}$$

Recent solution: $(0, 1, 0, 3, 1)$

Simplex Algorithm: Example I

Second tableau:

$$\begin{array}{rcrcrcr} x_2 & = & 1 & + & x_1 & - & x_3 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 1 & - & x_1 & + & x_3 \\ \hline z & = & 1 & + & 2x_1 & - & x_3 \end{array}$$

Only one candidate: x_1

$x_5 = 1 - x_1 + x_3$ is critical. Replace 5 by 1 in B: $B = \{1, 2, 4\}$.

$x_1 = 1 + x_3 - x_5$.

Third tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 1 & + & x_3 & - & x_5 \\ x_2 & = & 2 & & & - & x_5 \\ x_4 & = & 2 & - & x_3 & + & x_5 \\ \hline z & = & 3 & + & x_3 & - & 2x_5 \end{array}$$

Recent solution: $x = (1, 2, 0, 2, 0)$.

Simplex Algorithm: Example I

Third tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 1 & + & x_3 & - & x_5 \\ x_2 & = & 2 & & & - & x_5 \\ x_4 & = & 2 & - & x_3 & + & x_5 \\ \hline z & = & 3 & + & x_3 & - & 2x_5 \end{array}$$

Only one candidate: x_3
$x_4 = 2 - x_3 + x_5$ is critical. Replace 4 by 3 in B: $B = \{1, 2, 3\}$.
$x_3 = 2 - x_4 + x_5$
Fourth tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 3 & - & x_4 & & \\ x_2 & = & 2 & & & - & x_5 \\ x_3 & = & 2 & - & x_4 & + & x_5 \\ \hline z & = & 5 & - & x_4 & - & x_5 \end{array}$$

Recent solution: $x = (3, 2, 2, 0, 0)$.

Simplex Algorithm: Example I

Fourth tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 3 & - & x_4 & & \\ x_2 & = & 2 & & & - & x_5 \\ x_3 & = & 2 & - & x_4 & + & x_5 \\ \hline z & = & 5 & - & x_4 & - & x_5 \end{array}$$

Recent solution: $x = (3, 2, 2, 0, 0)$.

This is an optimum solution!

Second Example: Unboundedness

Simplex Algorithm: Example II: Unboundedness

$$
\begin{array}{rrrrrrrrrl}
\max & x_1 & & & & & & & & \\
s.t. & x_1 & - & x_2 & + & x_3 & & & = & 1 \\
 & -x_1 & + & x_2 & & & + & x_4 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Initial basis: B={3,4}
Simplex Tableau:

$$
\begin{array}{rrrrrrr}
x_3 & = & 1 & - & x_1 & + & x_2 \\
x_4 & = & 2 & + & x_1 & - & x_2 \\
\hline
z & = & & & x_1 & &
\end{array}
$$

Recent solution: $x = (0, 0, 1, 2)$.

Simplex Algorithm: Example II: Unboundedness

First Tableau:

$$\begin{array}{rcrcrcr} x_3 & = & 1 & - & x_1 & + & x_2 \\ x_4 & = & 2 & + & x_1 & - & x_2 \\ \hline z & = & & & x_1 & & \end{array}$$

Only one candidate: x_1. $x_3 = 1 - x_1 + x_2$ is critical. Replace 3 by 1 in B: $B = \{1, 4\}$.

$x_1 = 1 + x_2 - x_3$.

Second Tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 1 & + & x_2 & - & x_3 \\ x_4 & = & 3 & & & - & x_3 \\ \hline z & = & 1 & + & x_2 & - & x_3 \end{array}$$

Recent solution:

$x = (1, 0, 0, 3)$.

Simplex Algorithm: Example II: Unboundedness

Second Tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 1 & + & x_2 & - & x_3 \\ x_4 & = & 3 & & & - & x_3 \\ \hline z & = & 1 & + & x_2 & - & x_3 \end{array}$$

Only one candidate: x_2. No constraint for it!

$\Rightarrow$ The LP is unbounded

Second Example: Degeneracy

Simplex Algorithm: Example III: Degeneracy

$$
\begin{array}{lrcrcrcrcr}
\max & & & x_2 & & & & & & \\
s.t. & -x_1 & + & x_2 & + & x_3 & & & = & 0 \\
 & x_1 & & & & & + & x_4 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Initial basis: $B = \{3, 4\}$

Simplex Tableau:

$$
\begin{array}{rcrcrcr}
x_3 & = & & & x_1 & - & x_2 \\
x_4 & = & 2 & - & x_1 & & \\
\hline
z & = & & & & & x_2
\end{array}
$$

$\Rightarrow x = (0, 0, 0, 2)$: degenerated solution.

Simplex Algorithm: Example III: Degeneracy

First Tableau:

$$\begin{array}{rcrcrcr} x_3 & = & & & x_1 & - & x_2 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & & & & & x_2 \end{array}$$

Want to increase x_2. $x_3 = x_1 - x_2$ is critical. Replace 3 by 2 in B:
$B = \{2, 4\}$.
$x_2 = x_1 - x_3$. We will replace 3 by 2 in the basis.
But: We cannot increase x_2.
Second Tableau:

$$\begin{array}{rcrcrcr} x_2 & = & & & x_1 & - & x_3 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & & & x_1 & - & x_3 \end{array}$$

Recent solution: $x = (0, 0, 0, 2)$.

Simplex Algorithm: Example III: Degeneracy

Second Tableau:

$$
\begin{array}{rcrcrcr}
x_2 & = & & & x_1 & - & x_3 \\
x_4 & = & 2 & - & x_1 & & \\
\hline
z & = & & & x_1 & - & x_3
\end{array}
$$

Increase x_1. $x_4 = 2 - x_1$ is critical. $x_1 = 2 - x_4$. New base $B = \{1, 2, 0, 0\}$.

Third Tableau:

$$
\begin{array}{rcrcrcr}
x_1 & = & 2 & & & - & x_4 \\
x_2 & = & 2 & - & x_3 & - & x_3 \\
\hline
z & = & 2 & - & x_3 & - & x_4
\end{array}
$$

Optimum solution: $x = (2, 2, 0, 0)$.

The Simplex Algorithm

Algorithm 1: Simplex Algorithm

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$

Output: $\tilde{x} \in \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ maximizing $c^t x$ or the message that $\max\{c^t x \mid Ax = b, x \geq 0\}$ is unbounded or infeasible

1 Compute a feasible basis B;

2 If no such basis exists, stop with the message "INFEASIBLE";

3 Set $N = \{1, \ldots, n\} \setminus B$ and compute the feasible basic solution x for B;

4 Compute the simplex tableau $\begin{array}{ccccc} x_B & = & p & + & Qx_N \\ \hline z & = & z_0 & + & r^t x_N \end{array}$ for B;

5 **if** $r \leq 0$ **then**

  return $\tilde{x} = x$;

6 Choose $\alpha \in N$ with $r_\alpha > 0$;

7 **if** $q_{i\alpha} \geq 0$ **for all** $i \in B$ **then**

  return "UNBOUNDED";

8 Choose $\beta \in B$ with $q_{\beta\alpha} < 0$ and $\frac{p_\beta}{q_{\beta\alpha}} = \max\{\frac{p_i}{q_{i\alpha}} \mid q_{i\alpha} < 0, i \in B\}$;

9 Set $B = (B \setminus \{\beta\}) \cup \{\alpha\}$;

10 GOTO line 3;

Definition

Let G be an directed graph with capacities $u : E(G) \to \mathbb{R}_{>0}$ and numbers $b : V(G) \to \mathbb{R}$ with $\sum_{v \in V(G)} b(v) = 0$. A **feasible b-flow in (G, u, b)** is a mapping $f : E(G) \to \mathbb{R}_{\geq 0}$ with

- $f(e) \leq u(e)$ for all $e \in E(G)$ and
- $\sum_{e \in \delta_G^+(v)} f(e) - \sum_{e \in \delta_G^-(v)} f(e) = b(v)$ for all $v \in V(G)$.

Notation:

- $b(v)$: **balance** of v.
- If $b(v) > 0$, we call it the **supply** of v.
- If $b(v) < 0$, we call it the **demand** of v.
- Nodes v of G with $b(v) > 0$ are called **sources**.
- Nodes v with $b(v) < 0$ are called **sinks**.

Minimum-Cost Flow Problem

- **Input:** A directed graph G, capacities $u : E(G) \to \mathbb{R}_{>0}$, numbers $b : V(G) \to \mathbb{R}$ with $\sum_{v \in V(G)} b(v) = 0$, edge costs $c : E(G) \to \mathbb{R}$.
- **Task:** Find a b-flow f minimizing $\sum_{e \in E(G)} c(e) \cdot f(e)$.

Definition

Let G be a directed graph.

- For $e = (v, w)$ let $\overleftarrow{e} = (w, v)$ its **reverse edge**.
- Define $\overleftrightarrow{G}$ by $V(\overleftrightarrow{G}) = V(G)$ and $E(\overleftrightarrow{G}) = E(G) \dot\cup \{\overleftarrow{e} \mid e \in E(G)\}$.
- Edge costs $c : E(G) \to \mathbb{R}$ are extended to $E(\overleftrightarrow{G})$ by $c(\overleftarrow{e}) := -c(e)$.
- Let (G, u, b, c) be a MINIMUM-COST FLOW instance and let f be a b-flow in (G, u). The **residual graph** $G_{u,f}$ is defined by $V(G_{u,f}) := V(G)$ and
$$E(G_{u,f}) := \{e \in E(G) \mid f(e) < u(e)\} \quad \dot\cup \quad \{\overleftarrow{e} \in E(\overleftrightarrow{G}) \mid f(e) > 0\}.$$
- For $e \in E(G)$ we define the **residual capacity** by $u_f(e) = u(e) - f(e)$ and by $u_f(\overleftarrow{e}) = f(e)$.

Augmenting Flow

If P is a subgraph of the residual graph $G_{u,f}$ then **augmenting f along P by γ** (for $\gamma > 0$) means increasing P on forward edges in P (i.e. edges in $E(G) \cap E(P)$) by γ and reducing it on reverse edges in P by γ.

Algorithm 2: Network Simplex Algorithm

Input: A MIN-COST-FLOW instance (G, u, b, c);
A strongly feasible tree structure (r, T, L, U).

Output: A minimum-cost flow f.

1 Compute b-flow f and potential π associated to (r, T, L, U);
2 $e_0 :=$ an edge with $e_0 \in L$ and $c_\pi(e_0) < 0$ or with $e_0 \in U$ and $c_\pi(e_0) > 0$;
3 **if** (No such edge exists) **then return** f
4 $C :=$ the fund. circuit of e_0 (if $e_0 \in L$) or of $\overleftarrow{e_0}$ (if $e_0 \in U$) and let $\rho = c_\pi(e_0)$;
5 $\gamma := \min_{e' \in E(C)} u_f(e')$.
6 $e' :=$ last edge on C with $u_f(e') = \gamma$ when C is traversed starting at the peak;
7 Let e_1 be the corresponding edge in G, i.e. $e' = e_1$ or $e' = \overleftarrow{e_1}$;
8 Remove e_0 from L or U;
9 Set $T = (T \cup \{e_0\}) \setminus \{e_1\}$;
10 **if** $e' = e_1$ **then** Set $U = U \cup \{e_1\}$;
11 **else** Set $L = L \cup \{e_1\}$;
12 Augment f along γ by C;
13 Let X be the connected component of $(V(G), T \setminus \{e_0\})$ that contains r;
14 **if** $e_0 \in \delta^+(X)$ **then** Set $\pi(v) = \pi(v) + \rho$ for $v \in V(G) \setminus X$;
15 **if** $e_0 \in \delta^-(X)$ **then** Set $\pi(v) = \pi(v) - \rho$ for $v \in V(G) \setminus X$;
16 **go to** line 2;

Illustration:

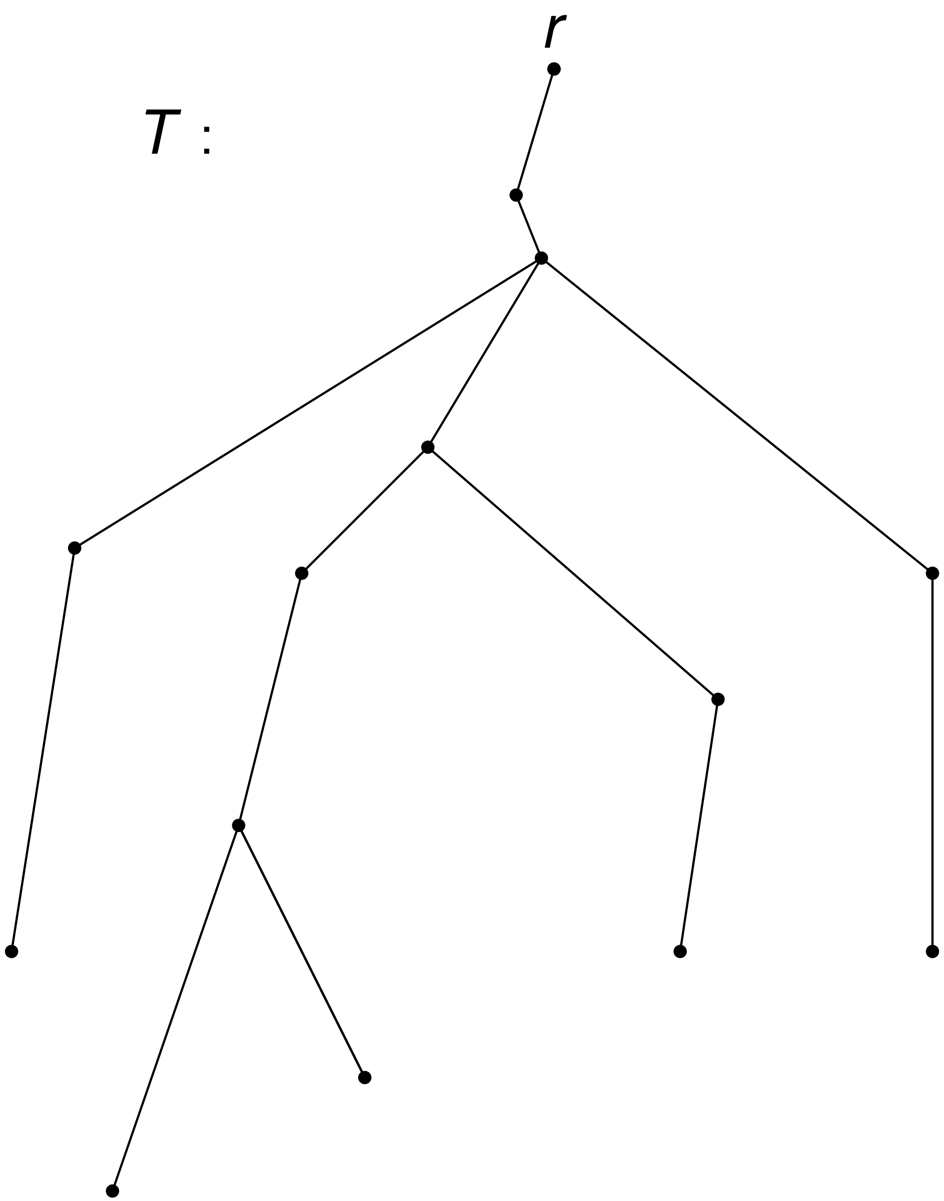
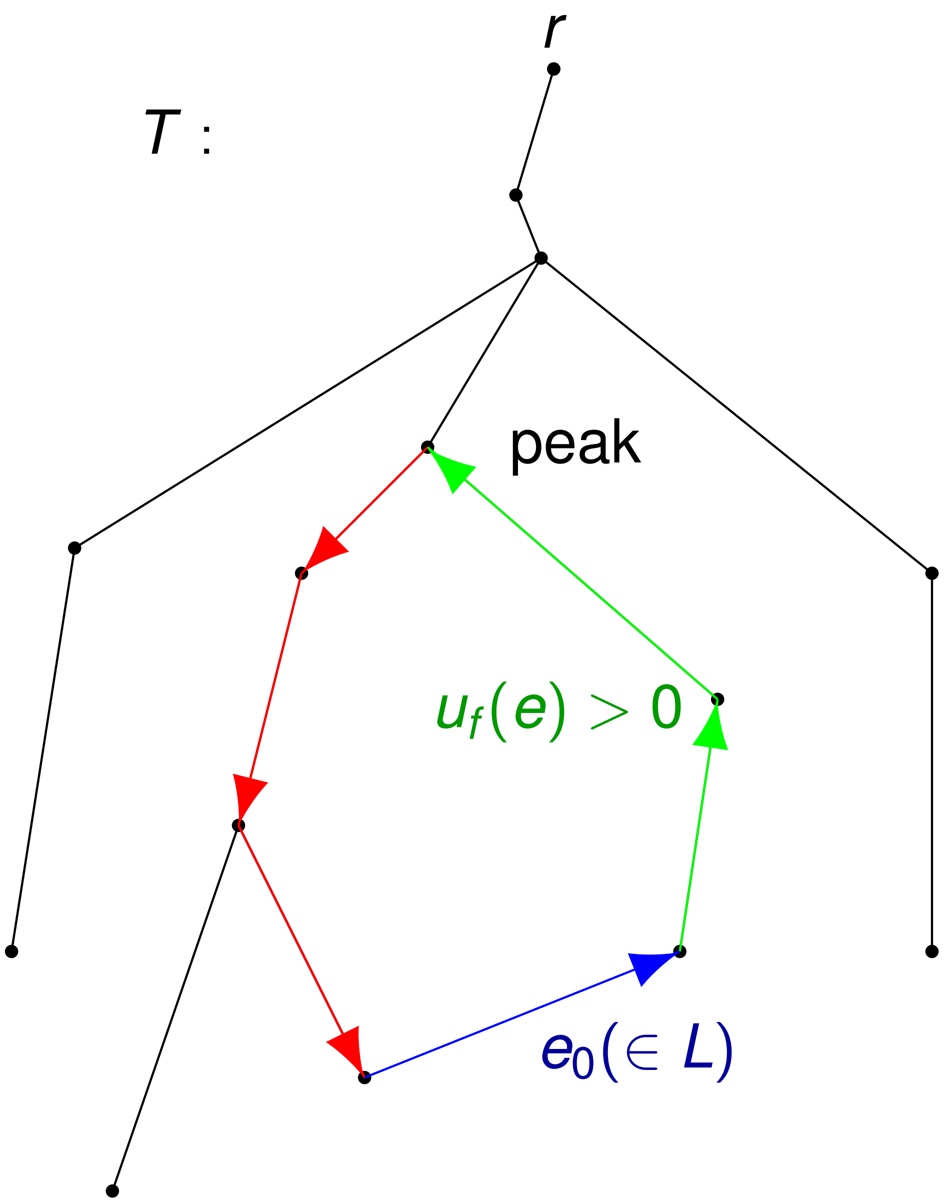
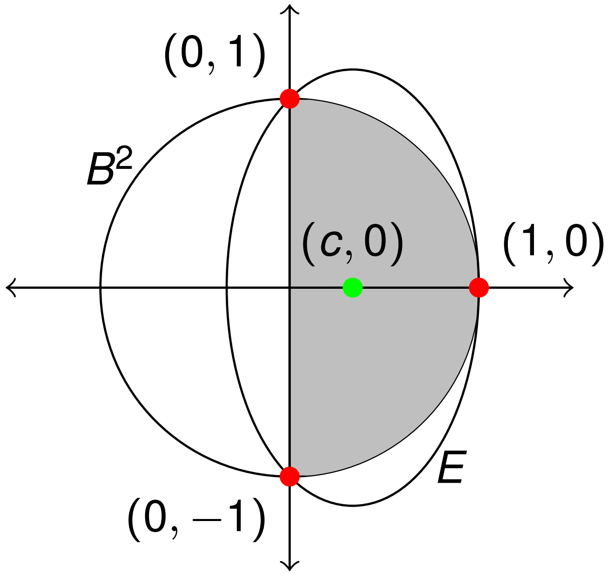


Illustration:



Cost of fundamental circuit = $c_\pi(e_0)$.



Half-Ball Lemma

$$B^n \cap \{x \in \mathbb{R}^n \mid x_1 \geq 0\} \quad \subseteq \quad E$$

with

$$E = \left\{ x \in \mathbb{R}^n \mid \frac{(n+1)^2}{n^2}\left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\sum_{i=2}^{n} x_i^2 \leq 1 \right\}.$$

Moreover, $\frac{\mathrm{vol}(E)}{\mathrm{vol}(B^n)} \leq e^{-\frac{1}{2(n+1)}}$.

Algorithm 3: Idealized Ellipsoid Algorithm

Input: A separation oracle for a closed convex set $K \subseteq \mathbb{R}^n$, a number $R > 0$ with $K \subseteq \{x \in \mathbb{R}^n \mid x^t x \leq R^2\}$, and a number $\epsilon > 0$.

Output: An $x \in K$ or the message "$\text{vol}(K) < \epsilon$".

$p_0 := 0$, $A_0 := R^2 I_n$;

for $k = 0, \ldots, N(R, \epsilon) := \lfloor 2(n+1)(n \ln(2R) + \ln(\frac{1}{\epsilon})) \rfloor$ **do**

  if $p_k \in K$ **then**

    return p_k;

  Let $\bar{a} \in \mathbb{R}^n$ be a vector with $\bar{a}^t y \geq \bar{a}^t p_k$ for all $y \in K$;

  $b_k := \frac{A_k \bar{a}}{\sqrt{\bar{a}^t A_k \bar{a}}}$;

  $p_{k+1} := p_k + \frac{1}{n+1} b_k$;

  $A_{k+1} := \frac{n^2}{n^2-1}(A_k - \frac{2}{n+1} b_k b_k^t)$;

return "$\text{vol}(K) < \epsilon$";

$\widetilde{p_k}$ and $\widetilde{A_k}$: exact values
p_k and A_k: rounded values

$x \in K$:

- $(x - \widetilde{p}_k)^t \widetilde{A_k}^{-1} (x - \widetilde{p}_k) \leq 1$
- $(x - p_k)^t A_k^{-1} (x - p_k) \quad \leq \quad 1 + 2\sqrt{n}\delta \, \|\widetilde{A_k}^{-1}\| \, (R + \|\widetilde{p_k}\|) + n\delta^2 \|\widetilde{A_k}^{-1}\| + (R + \|p_k\|)^2 \, \|A_k^{-1}\| \cdot \|\widetilde{A_k}^{-1}\| \cdot n\delta$

Adjust $\widetilde{A_k}$ by multiplying it by $\mu = 1 + \frac{1}{2n(n+1)}$. $\Rightarrow$

$$(x - \widetilde{p}_k)^t \widetilde{A_k}^{-1} (x - \widetilde{p}_k) = \frac{1}{1 + \frac{1}{2n(n+1)}} < 1 - \frac{1}{4n^2}.$$

- $(x - \widetilde{p}_k)^t \widetilde{A_k}^{-1} (x - \widetilde{p}_k) \quad \leq \quad 1 - \frac{1}{4n^2}$
- $(x - p_k)^t A_k^{-1} (x - p_k) \quad \leq \quad 1 - \frac{1}{4n^2} + 2\sqrt{n}\delta \ \|\widetilde{A_k}^{-1}\| \ (R + \|\widetilde{p}_k\|) + n\delta^2 \|\widetilde{A_k}^{-1}\| + (R + \|p_k\|)^2 \ \|A_k^{-1}\| \cdot \|\widetilde{A_k}^{-1}\| \cdot n\delta$

Goal is to choose δ such that

- $2\sqrt{n}\delta \ \|\widetilde{A_k}^{-1}\| \ (R + \|\widetilde{p}_k\|) + n\delta^2 \|\widetilde{A_k}^{-1}\| \ + \ (R + \|p_k\|)^2 \ \|A_k^{-1}\| \cdot \|\widetilde{A_k}^{-1}\| n\delta < \frac{1}{4n^2}$
- $\delta \|\widetilde{A_{k+1}}^{-1}\| < \frac{1}{4(n+1)^3}$

Algorithm 4: Ellipsoid Algorithm

Input: A separation oracle for a closed convex set $K \subseteq \mathbb{R}^n$, a number $R > 0$ with $K \subseteq \{x \in \mathbb{R}^n \mid x^t x \leq R^2\}$, and a number $\epsilon > 0$

Output: An $x \in K$ or the message "$\text{vol}(K) < \epsilon$".

$p_0 := 0$, $A_0 := R^2 I_n$;

for $k = 0, \ldots, N(R, \epsilon) := \lceil 8(n+1)(n \ln(2R) + \ln(\frac{1}{\epsilon})) \rceil$ **do**

  if $p_k \in K$ **then**

    return p_k;

  Let $\bar{a} \in \mathbb{R}^n$ be a vector with $\bar{a}^t y \geq \bar{a}^t p_k$ for all $y \in K$;

  $b_k := \frac{A_k \bar{a}}{\sqrt{\bar{a}^t A_k \bar{a}}}$;

  p_{k+1} an approximation of $\widetilde{p_{k+1}} := p_k + \frac{1}{n+1} b_k$ with maximum error $\delta < \left(2^{6(N(R,\epsilon)+1)} 16 n^3\right)^{-1}$;

  A_{k+1} a symmetric approximation of $\widetilde{A_{k+1}} := \left(1 + \frac{1}{2n(n+1)}\right) \frac{n^2}{n^2-1} \left(A_k - \frac{2}{n+1} b_k b_k^t\right)$ with maximum error δ;

return "$\text{vol}(K) < \epsilon$";

Let $P \subseteq \mathbb{R}^n$ be a rational polytope and let $x_0 \in P$ in the interior of P. Let $T \in \mathbb{N} \setminus \{0\}$ such that $\text{size}(x_0) \leq \log(T)$ and $\text{size}(x) \leq \log(T)$ for all vertices x of P.

Theorem (Separation $\rightarrow$ Optimization)

Let $c \in \mathbb{Q}^n$. Given n, c, x_0, T and a polynomial-time separation oracle for P, a vertex x^* of P attaining $\max\{c^t x \mid x \in P\}$ can be found in time polynomial in n, $\log(T)$ and $\text{size}(c)$.

Theorem (Optimization $\rightarrow$ Separation)

Let $y \in \mathbb{Q}^n$. Given n, y, x_0, T and an oracle which for given $c \in \mathbb{Q}^n$ returns a vertex x^* of P attaining $\max\{c^t x \mid x \in P\}$, we can implement a separation oracle for P and y with running time polynomial in n, $\log(T)$ and $\text{size}(y)$. If $y \notin P$, we can find with this running time a facet-defining inequality of P that is violated by y.

Interior Point Methods

Primal-dual pair:

Primal:

$$\begin{aligned} & \max c^t x \\ \text{s.t.} \quad & Ax + s = b \\ & s \geq 0 \end{aligned} \tag{1}$$

Dual:

$$\begin{aligned} & \min b^t y \\ \text{s.t.} \quad & A^t y = c \\ & y \geq 0 \end{aligned} \tag{2}$$

We want to compute a solution of the dual LP.

May assume:

- Columns of A are linearly independent
- More rows than columns

Combined constraints:

$$\begin{aligned} Ax + s &= b \\ A^t y &= c \\ y^t s &= 0 \\ y &\geq 0 \\ s &\geq 0 \end{aligned} \tag{3}$$

New set of constraints:

$$\begin{aligned} Ax + s &= b \\ A^t y &= c \\ \sum_{i=1}^{m} \left(\frac{y_i s_i}{\mu} - 1 \right)^2 &\leq \frac{1}{4} \\ y &> 0 \\ s &> 0 \end{aligned} \tag{4}$$

General strategy:

(I) Compute an initial solution of a modified version of (4): ✓

(II) Reduce μ by a constant factor and adapt x, y and s to the new value of μ such that we again get a solution of (4).
Iterate this step until μ is small enough.

(III) Compute an optimum solution of the dual LP.

Assumption: We have a solution $\mu^{(k)}, x^{(k)}, y^{(k)}, s^{(k)}$ of the system

$$\begin{array}{rcl} \tilde{A}x + s &=& \tilde{b} \\ \tilde{A}^t y &=& \tilde{c} \\ \sum_{i=1}^{m+2} \left(\frac{y_i s_i}{\mu} - 1 \right)^2 &\leq& \frac{1}{4} \\ y &>& 0 \\ s &>& 0 \end{array}$$

Goal: Find solution $\mu^{(k+1)}, x^{(k+1)}, y^{(k+1)}, s^{(k+1)}$ with $\mu^{(k+1)} = (1-\delta)\mu^{(k)}$ ($\delta \in (0,1)$ will be defined later).

Notation:

- $x^{(k+1)} = x^{(k)} + f$
- $y^{(k+1)} = y^{(k)} + g$
- $s^{(k+1)} = s^{(k)} + h$

$\Rightarrow$

- $\tilde{A}^t g = 0$
- $\tilde{A} f + h = 0$

We want $y_i^{(k+1)} s_i^{(k+1)}$ to be close to $\mu^{(k+1)}$.

We have

$$
\begin{aligned}
y_i^{(k+1)} s_i^{(k+1)} &= (y_i^{(k)} + g_i)(s_i^{(k)} + h_i) \\
&= y_i^{(k)} s_i^{(k)} + g_i s_i^{(k)} + y_i^{(k)} h_i + g_i h_i
\end{aligned}
$$

We demand $y_i^{(k)} s_i^{(k)} + g_i s_i^{(k)} + y_i^{(k)} h_i = \mu^{(k+1)}$

Equation system:

$$
\begin{aligned}
\tilde{A}^t g &= 0 \\
\tilde{A} f + h &= 0 \\
s_i^{(k)} g_i + y_i^{(k)} h_i &= \mu^{(k+1)} - y_i^{(k)} s_i^{(k)} \qquad i = 1, \ldots, m+2
\end{aligned}
\qquad (*)
$$

Proposition

If $Ax \le b, a^t x \le \beta$ is TDI with a integral, then $Ax \le b, a^t x = \beta$ is also TDI.

Proof: Let c be an integral vector for which

$$\begin{aligned} & \max\{c^t x \mid Ax \le b, a^t x = \beta\} \\ = \quad & \min\{b^t y + \beta(\lambda - \mu) \mid y \ge 0, \lambda, \mu \ge 0, A^t y + (\lambda - \mu)a = c\} \end{aligned} \tag{5}$$

is finite. Let $x^*, y^*, \lambda^*, \mu^*$ be optimum primal and dual solutions. Set $\tilde{c} := c + \lceil \mu^* \rceil a$. Then,

$$\begin{aligned} & \max\{\tilde{c}^t x \mid Ax \le b, a^t x \le \beta\} \\ = \quad & \min\{b^t y + \beta\lambda \mid y \ge 0, \lambda \ge 0, A^t y + \lambda a = \tilde{c}\} \end{aligned} \tag{6}$$

is finite because x^* is feasible for the maximum and y^* and $\lambda^* + \lceil \mu^* \rceil - \mu^*$ are feasible for the minimum.

...

Theorem

For each rational polyhedron $P \subseteq \mathbb{R}^n$ there exists a rational TDI-system $Ax \leq b$ with $A \in \mathbb{Z}^{m \times n}$ and $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. The vector b can be chosen to be integral if and only if P is integral.

Proof: W.l.o.g. $P \neq \emptyset$. For each minimal face F of P, define

$$C_F := \{c \in \mathbb{R}^n \mid c^t z = \max\{c^t x \mid x \in P\} \text{ for all } z \in F\}.$$

Then, C_F is a polyhedral cone. To see this, let $P = \{\tilde{A}x \leq \tilde{b}\}$ be some description of P. Then C_F is generated by the rows of $\tilde{A}$ that are active in F.

Let F be a minimal face, and let $a_1, \dots, a_t$ be a Hilbert basis generating C_F. Choose $x_0 \in F$, and define $\beta_i := a_i^t x_0$ for $i = 1, \dots, t$. Then, $\beta_i = \max\{a_i^t x \mid x \in P\}$ $(i = 1, \dots, t)$. Let $\mathcal{S}_F$ be the system $a_1^t x \leq \beta_1, \dots, a_t^t x \leq \beta_t$. All inequalities in $\mathcal{S}_F$ are valid for P. Let $Ax \leq b$ be the union of the systems $\mathcal{S}_F$ over all minimal faces F of P. Then, $P \subseteq \{x \in \mathbb{R}^n \mid Ax \leq b\}$.

Theorem

A matrix $A = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \in \mathbb{Z}^{m\times n}$ is totally unimodular if and only if for each set $R \subseteq \{1,\ldots,n\}$ there is a partition $R = R_1 \dot\cup R_2$ such that for each $i \in \{1,\ldots,m\}$: $\sum_{j\in R_1} a_{ij} - \sum_{j\in R_2} a_{ij} \in \{-1,0,1\}$.

Proof: "$\Rightarrow$:" $\checkmark$

"$\Leftarrow$:" Assume: For each $R \subseteq \{1,\ldots,n\}$ there is a partition $R = R_1 \dot\cup R_2$ as above.

By induction in k: Every $k \times k$-submatrix of A has determinant -1,0, or 1.

$k = 1$: $\checkmark$

Let $k > 1$. Let $B = (b_{ij})_{i,j\in\{1,\ldots,k\}}$ a submatrix of A. W.l.o.g.: B is regular.

We have proved: $B^* := (\det(B))B^{-1} \in \{-1,0,1\}^{k\times k}$.

b^*: first column of B^*. Then, $Bb^* = \det(B)e_1$. Let $R := \{j \in \{1,\ldots,k\} \mid b^*_j \neq 0\}$. For $i \in \{2,\ldots,k\}$, we have $0 = (Bb^*)_i = \sum_{j\in R} b_{ij}b^*_j$, so $|\{j \in R \mid b_{ij} \neq 0\}|$ is even.

Let $R = R_1 \dot\cup R_2$ such that $\sum_{j\in R_1} b_{ij} - \sum_{j\in R_2} b_{ij} \in \{-1,0,1\}$ for all $i \in \{1,\ldots,k\}$. Thus, for $i \in \{2,\ldots,k\}$, we have: $\sum_{j\in R_1} b_{ij} - \sum_{j\in R_2} b_{ij} = 0$.

The **incidence matrix** of an undirected graph G is the matrix $A_G = (a_{v,e})_{\substack{v \in V(G) \\ e \in E(G)}}$ which is defined by:

$$a_{v,e} = \begin{cases} 1, & \text{if } v \in e \\ 0, & \text{if } v \notin e \end{cases}$$

The **incidence matrix** of a directed graph G is the matrix $A_G = (a_{v,e})_{\substack{v \in V(G) \\ e \in E(G)}}$ which is defined by:

$$a_{v,(x,y)} = \begin{cases} -1, & \text{if } v = x \\ 1, & \text{if } v = y \\ 0, & \text{if } v \notin \{x, y\} \end{cases}$$

Theorem:

For every rational polyhedron P, there is a number t with $P^{(t)} = P_I$.

Proof: Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with A integral and b rational. We prove the statement by induction on $n + \dim(P)$. The case $\dim(P) = 0$ is trivial.

Case 1: $\dim(P) < n$: $\checkmark$

Case 2: $\dim(P) = n$:

P is rational $\Rightarrow$ P_I is rational $\Rightarrow$ $P_I = \{x \in \mathbb{R}^n \mid Cx \leq d\}$ with C integral and d rational. If $P_I = \emptyset$, we choose $C = A$ and $d = b - A'\mathbb{1}_n$ where A' arises from A by taking the absolute value of each entry.

Let $c^t x \leq \delta$ be an inequality in $Cx \leq d$.

Claim: There is an $s \in \mathbb{N}$ with $P^{(s)} \subseteq H := \{x \in \mathbb{R}^n \mid c^t x \leq \delta\}$.

Proof of the claim: There is a $\beta \geq \delta$ with $P \subseteq \{x \in \mathbb{R}^n \mid c^t x \leq \beta\}$: If $P_I = \emptyset$, this is true by construction. If $P_I \neq \emptyset$, it follows from the fact that $c^t x$ is bounded over P if and only if it is bounded over P_I.

Algorithm 5: Branch-and-Bound Algorithm

Input: A matrix $A \in \mathbb{Q}^{m \times n}$, a vector $b \in \mathbb{Q}^m$, and a vector $c \in \mathbb{Q}^n$ such that the LP $\max\{c^t x \mid Ax \leq b\}$ is feasible and bounded.

Output: A vector $\tilde{x} \in \{x \in \mathbb{Z}^n \mid Ax \leq b\}$ maximizing $c^t x$ or the message that there is no feasible solution.

$L := -\infty$; $P_0 := \{x \in \mathbb{R}^n \mid Ax \leq b\}$; $\mathcal{K} := \{P_0\}$;
while $\mathcal{K} \neq \emptyset$ **do**
    Choose a $P_j \in \mathcal{K}$; $\mathcal{K} := \mathcal{K} \setminus \{P_j\}$;
    if $P_j \neq \emptyset$ **then**
        Let x^* be an optimum solution of $\max\{c^t x \mid x \in P_j\}$ and let $c^* = c^t x^*$;
        if $c^* > L$ **then**
            if $x^* \in \mathbb{Z}^n$ **then**
                $L := c^*$;
                $\tilde{x} := x^*$;
            else
                Choose $i \in \{1, \ldots, n\}$ with $x_i^* \notin \mathbb{Z}$;
                $P_{2j+1} := \{x \in P_j \mid x_i \leq \lfloor x_i^* \rfloor\}$;
                $P_{2j+2} := \{x \in P_j \mid x_i \geq \lceil x_i^* \rceil\}$;
                $\mathcal{K} := \mathcal{K} \cup \{P_{2j+1}\} \cup \{P_{2j+2}\}$;
if $L > -\infty$ **then**
    return $\tilde{x}$;
else
    return “There is no feasible solution”;

Branch-and-Bound: Example

$$
\begin{aligned}
\max \quad & -x_1 + 3x_2 \\
\text{subject to} \quad & -4x_1 + 6x_2 \leq 9 \\
& x_1 + x_2 \leq 4 \\
& x_1, x_2 \geq 0 \\
& x_1, x_2 \in \mathbb{Z}
\end{aligned}
$$

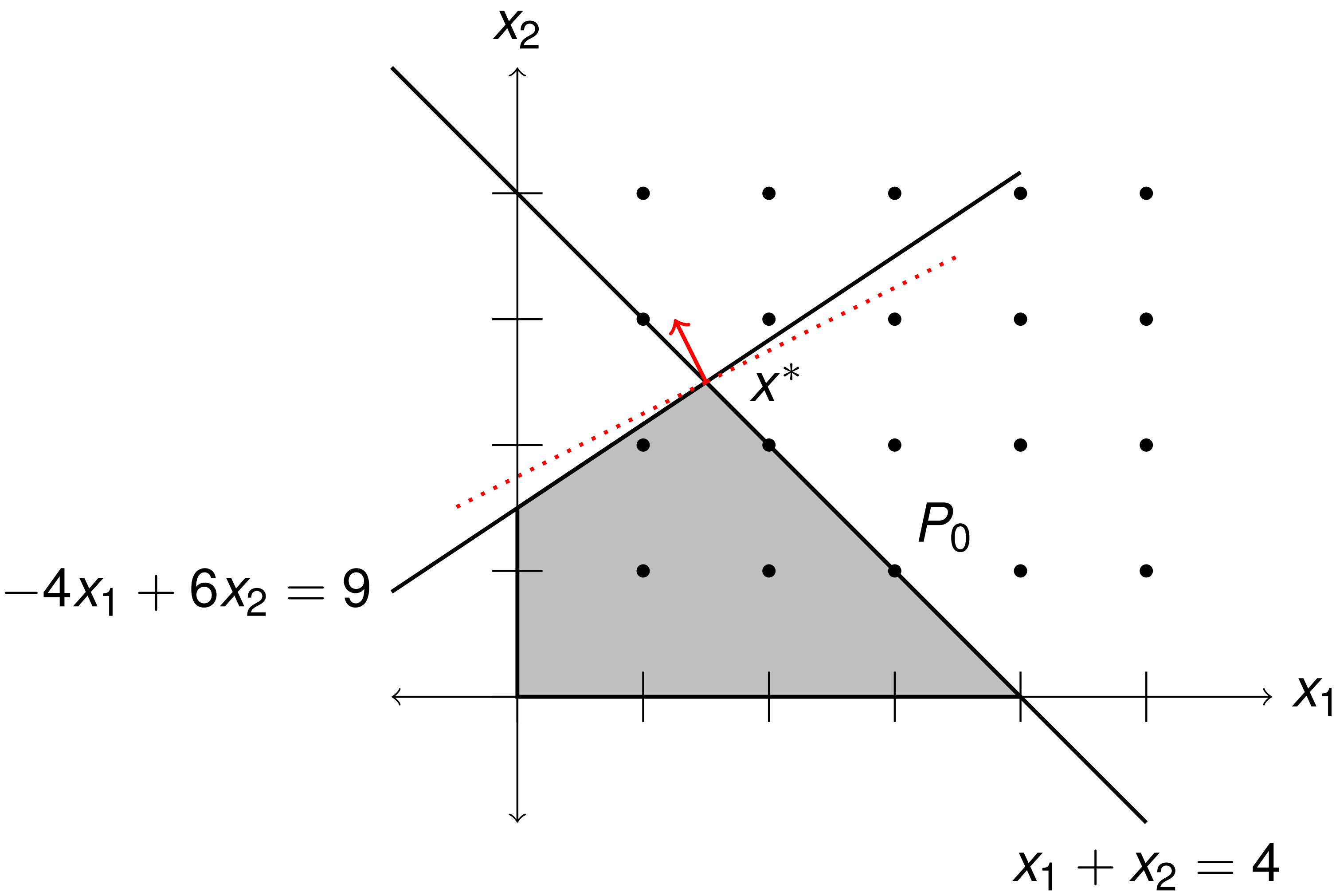


Figure : A branch-and-bound example (I).

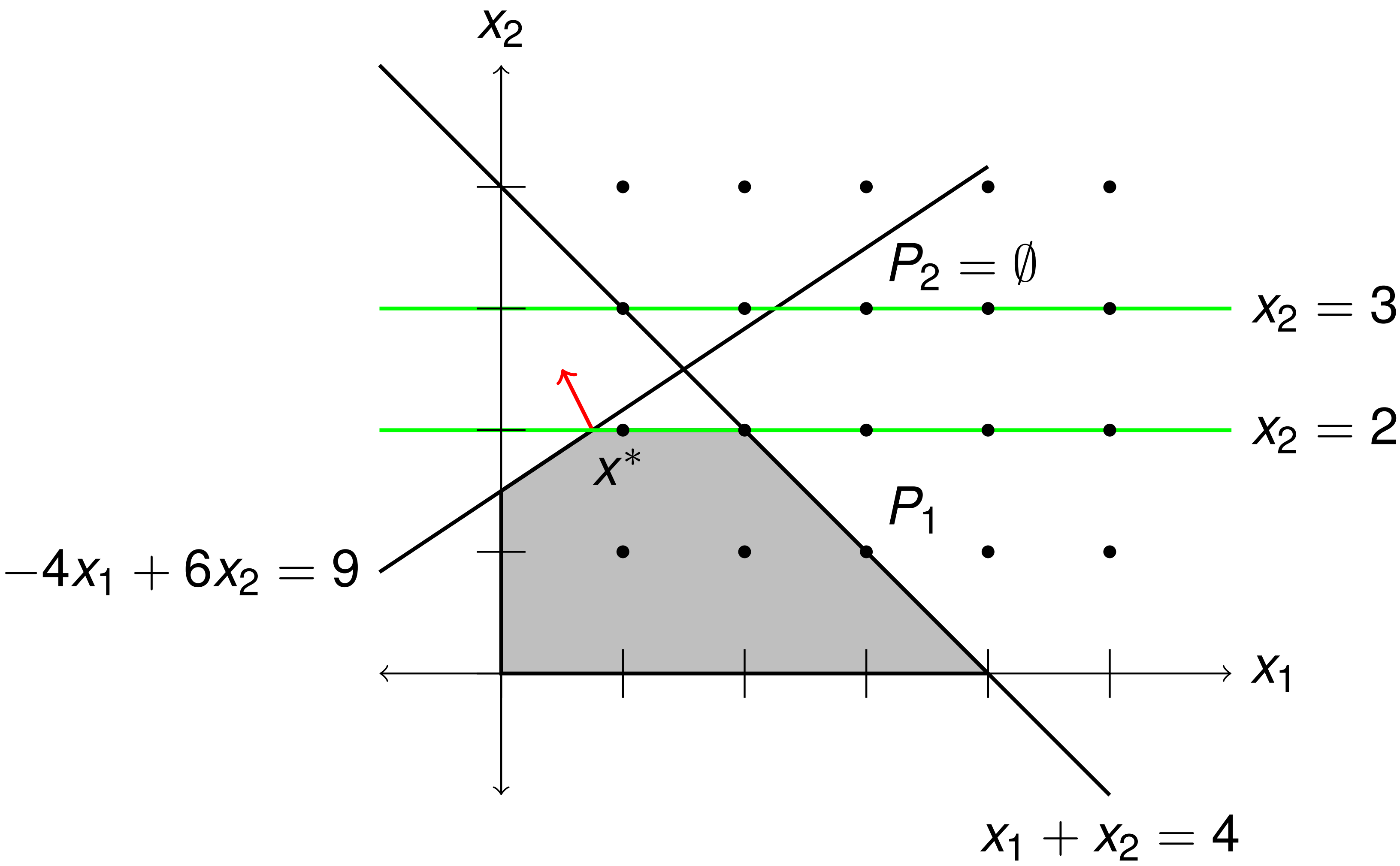


Figure : A branch-and-bound example (II).

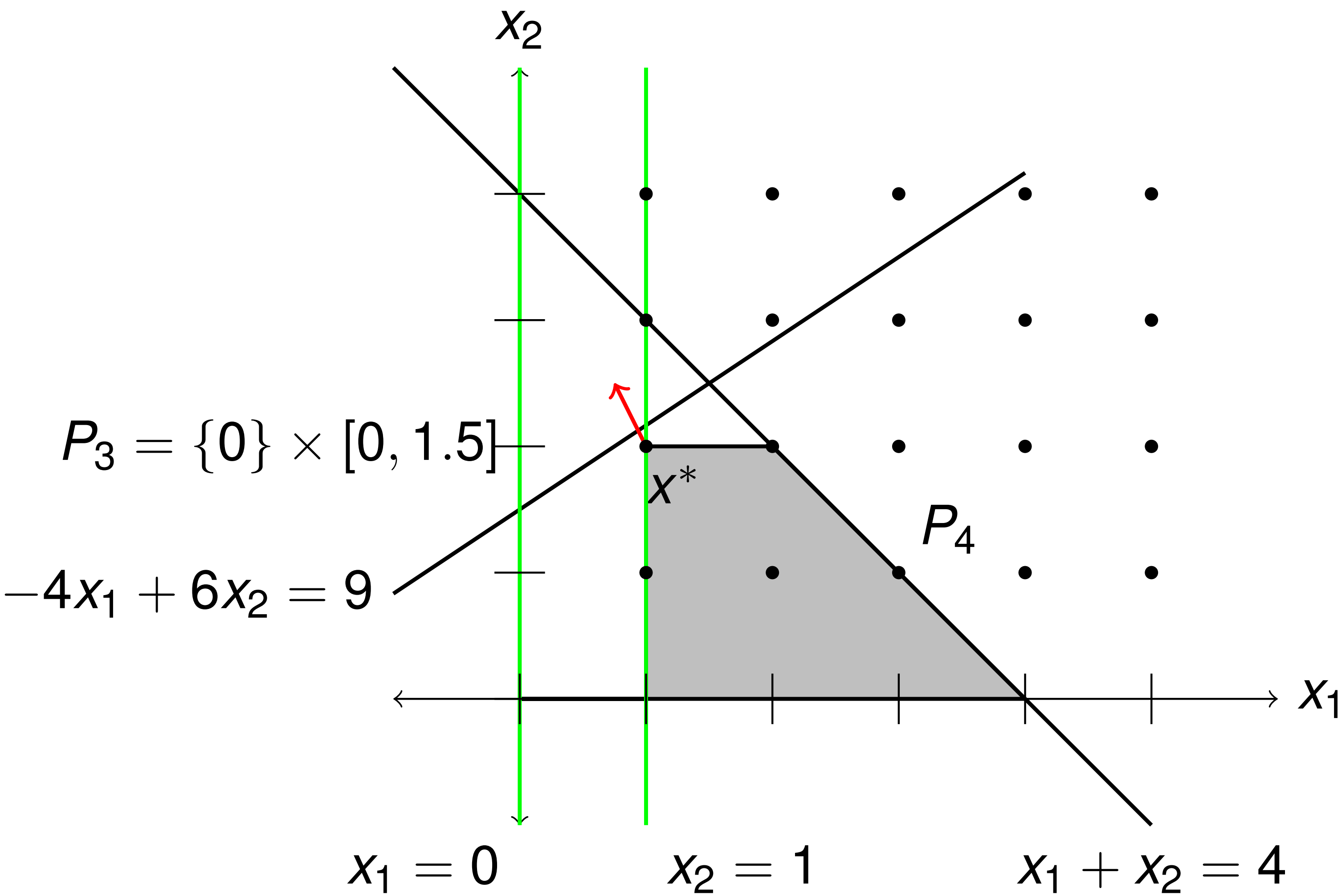


Figure : A branch-and-bound example (III).