Winter term 2013/14 Research Institute for Discrete Mathematics
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Combinatorial Optimization

Exercise Sheet 5

Exercise 5.1:
Let G be a graph and M a matching in G that is not maximum.

(i) Show that there are $\nu(G) - |M|$ vertex-disjoint M-augmenting paths in G.
Hint: Recall the proof of Berge's Theorem.

(ii) Prove that there exists an M-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

(iii) Let P be a shortest M-augmenting path in G and P' an $(M \triangle E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + 2|E(P \cap P')|$.

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then P_i and P_j are vertex-disjoint.

(v) Conclude that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains at most $2\sqrt{\nu(G)} + 2$ different numbers.

(6 points)

Exercise 5.2:
Let G be a k-regular and $(k-1)$-edge-connected graph with an even number of vertices. Let $c : E(G) \to \mathbb{R}_+$. Prove that there is a perfect matching M in G with $c(M) \geq \frac{1}{k} c(E(G))$.
Hint: Show that $\frac{1}{k}\mathbb{1}$ is in the perfect matching polytope where $\mathbb{1}$ denotes a vector whose components are all 1.

(4 points)

Exercise 5.3:
Let G be a graph and P the fractional perfect matching polytope of G. Prove that the vertices of P are exactly the vectors x with

$$x_e = \begin{cases} \frac{1}{2} & \text{if } e \in E(C_1) \cup \ldots \cup E(C_k) \\ 1 & \text{if } e \in M \\ 0 & \text{otherwise,} \end{cases}$$

where $C_1, \ldots, C_k$ are vertex-disjoint odd cycles and M is a perfect matching in $G - (V(C_1) \cup \ldots \cup V(C_k))$.

(4 points)

Exercise 5.4:
Show that Theorem 41 implies the Berge-Tutte-formula (Theorem 17).

(2 points)

Deadline: Thursday, November 21, 2013, before the lecture.