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Exercises 3

Exercise 1:
Let G be a bipartite graph with bipartition $V(G) = A \dot{\cup} B, A = \{a_1, \dots, a_k\}, B = \{b_1, \dots, b_k\}$. For any vector $x = (x_e)_{e \in E(G)}$ we define a matrix $M_G(x) = (m^x_{ij})_{1 \le i,j \le k}$ by

$$m^x_{ij} := \begin{cases} x_e & \text{if } e = \{a_i, b_j\} \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Its determinant $\det M_G(x)$ is a polynomial in $x = (x_e)_{e \in E(G)}$. Prove that G has a perfect matching if and only if $\det M_G(x)$ is not identically zero. (4 points)

Exercise 2:
Let G be a graph and M a matching in G that is not maximum.

(a) Show that there are $\nu(G) - |M|$ vertex-disjoint M-augmenting paths in G.
Hint: Recall the proof of Berge's Theorem (Thm. 6).

(b) Prove that there exists an M-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

(c) Let P be a shortest M-augmenting path in G, and P' an $(M \triangle E(P))$-augmenting path. Then $|E(P')| \ge |E(P)| + |E(P \cap P')|$.

Consider the following generic algorithm. We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \dots$ be the sequence of augmenting paths chosen. By (c), $|E(P_k)| \le |E(P_{k+1})|$ for all k.

(d) Show that if $|E(P_i)| = |E(P_j)|$ for $i \ne j$ then P_i and P_j are vertex-disjoint.

(e) Conclude that the sequence $|E(P_1)|, |E(P_2)|, \dots$ contains at most $2\sqrt{\nu(G)} + 2$ different numbers.

(6 points)

Exercise 3:
For a graph G, let $\mathcal{T}(G) := \{X \subseteq V(G) | q_g(X) > |X|\}$ the familly of Tutte-sets of G. Prove or find a counterexample: G is factor-critical if and only if $\mathcal{T}(G) = \{\emptyset\}$.

(4 points)

Exercise 4: Prove:

1. An undirected graph G is 2-edge-connected if and only if $|E(G)| \ge 2$ and G has an ear-decomposition.

2. A directed graph is strongly connected if and only if it has an ear-decomposition.

(4 points)

Deadline: Tuesday, November 2nd, before the lecture.