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Exercises 2

Exercise 1:
Let $\alpha(G)$ denote the size of a maximum stable set in a graph G, and $\zeta(G)$ the minimum cardinality of an edge cover.
(An *edge cover* in G is a set $F \subseteq E(G)$ of edges such that every vertex of G is incident to at least one edge in F.)
Prove:

(a) $\alpha(G) + \tau(G) = |V(G)|$ for any graph G.

(b) $\nu(G) + \zeta(G) = |V(G)|$ for any graph G with no isolated vertices.

(c) $\zeta(G) = \alpha(G)$ for any bipartite graph G with no isolated vertices.

(1 + 2 + 1 points)

Exercise 2:

(a) Let $S = \{1, 2, \ldots, n\}$ and $0 \leq k < \frac{n}{2}$. Let A and B be the collection of all k-element and $(k+1)$-element subsets of S, respectively. Construct a bipartite graph
$$G = (A \,\dot{\cup}\, B, \{\{a, b\} : a \in A,\ b \in B,\ a \subseteq b\}).$$
Prove that G has a matching covering A.

(b) Prove Sperner's Lemma: the maximum number of subsets of an n-element set such that none is contained in any other is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

(2 + 2 points)

Exercise 3:
Let G be a bipartite graph with bipartition $V(G) = A \,\dot{\cup}\, B$. Suppose that $S \subseteq A$, $T \subseteq B$, and there is a matching covering S and a matching covering T. Prove that then there is a matching covering $S \cup T$.

(4 points)

Exercise 4:
Prove that every 3-regular simple graph with at most two bridges has a perfect matching. Is there a 3-regular simple graph without a perfect matching?
Hint: Use Tutte's Theorem 10.13.

(4 points)

Deadline: Tuesday, October 26th, before the lecture.