Prof. Dr. Stephan Held
Paula Heinz, M. Sc.

Exercise Set 5

Exercise 5.1. Let (M, d) be a metric space. For $n \in \mathbb{N}_{\geq 2}$ we define the Steiner ratios

$$\mathrm{SR}(M, n) := \sup_{P=\{p_1,\ldots,p_n\}\subseteq M} \frac{\mathrm{MST}(P)}{\mathrm{STEINER}(P)},$$

where $\mathrm{STEINER}(P)$ denotes the length of a minimum Steiner tree for P and $\mathrm{MST}(P)$ denotes the size of a minimum spanning for P.

Let $(M, d) = (\mathbb{R}^2, d)$ with $d(x, y) := ||x - y||_2$. Show that there is a $n \in \mathbb{N}_{\geq 2}$, such that $\mathrm{SR}(M, n) \geq 2/\sqrt{3}$.

(3 points)

Exercise 5.2. Let G be an undirected connected graph with edge weights $c : E(G) \to \mathbb{R}_+$. Let $\emptyset \neq L \subseteq V(G)$.
Let $s, t \in V(G)$ and $\pi(v) := \max_{l\in L}\{|\mathrm{dist}_{(G,c)}(t, l) - \mathrm{dist}_{(G,c)}(v, l)|\}$ for $v \in V(G)$.
Define a new graph G' with

$$V(G') := V(G) \qquad\qquad E(G') := \{(x, y), (y, x) : \{x, y\} \in E(G)\}$$

The *reduced costs* $c_\pi : E(G') \to \mathbb{R}$ are defined as

$$c_\pi((x, y)) := c(\{x, y\}) - \pi(x) + \pi(y).$$

Show:

(a) $c_\pi(e) \geq 0 \ \forall e \in E(G')$

(b) Any shortest s-t-path in (G', c_π) is a shortest s-t-path in (G, c)

(c) $\{v \in V(G) : \mathrm{dist}_{(G',c_\pi)}(s, v) < \mathrm{dist}_{(G',c_\pi)}(s, t)\} \subseteq \{v \in V(G) : \mathrm{dist}_{(G,c)}(s, v) < \mathrm{dist}_{(G,c)}(s, t)\}$

Remark: If many shortest paths need to be computed in a graph, it might be worthwile to precompute distances to so called *landmarks* L, to then do a faster pathsearch in (G, c_π).

(3 + 1 + 2 points)

Exercise 5.3. Consider the following Clustered Rectilinear Steiner Tree Problem: Given a partition $T = \dot{\bigcup}_{i=1}^{k} P_i$ of the terminals ($\emptyset \neq P_i \subseteq \mathbb{R}^2, |P_i| < \infty$), find a (rectilinear) Steiner tree Y_i for each set of terminals P_i and one rectilinear, toplevel (group) Steiner tree Y_{top} connecting the embedded trees Y_i ($i = 1, \dots, k$). The task is to minimize the total length of all trees.

Let A be an α-approximation algorithm for the Rectilinear Steiner Tree Problem. A feasible solution to the Clustered Rectilinear Steiner Tree Problem can be found by first selecting a connection point $q_i \in \mathbb{R}^2$ for each $i = 1, \dots, k$ and then computing $Y_i := A(P_i \cup \{q_i\})$ and $Y_{\text{top}} := A(\{q_i : 1 \leq i \leq n\})$.

(a) Show that picking $q_i \in P_i$ arbitrarily yields a 2α approximation.

(b) Prove that choosing each q_i as the center of the bounding box of P_i implies a $\frac{7}{4}\alpha$ approximation algorithm.

(2 + 4 points)

Exercise 5.4. Let N be a finite set of pins, and let S_p be a set of axis-parallel rectangles for each $p \in N$. We want to compute the *bounding box netlength* of N, i.e. an axis-parallel rectangle R with minimum perimeter s.t. for every $p \in N$ there is an $S \in S_p$ with $R \cap S \neq \emptyset$.

Show how to compute such a rectangle in $O(n^3)$ time where $n := \sum_{p \in N} |S_p|$.
(5 points)

Deadline: May 20th, before the lecture. The websites for lecture and exercises can be found at:

`https://www.or.uni-bonn.de/lectures/ss25/chipss25_ex.html`

In case of any questions feel free to contact me at heinz@dm.uni-bonn.de.