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Exercise Set 4

Exercise 4.1. For a finite set $\emptyset \neq T \subsetneq \mathbb{R}^2$ we define

$$\mathrm{BB}(T) := \max_{(x,y)\in T} x - \min_{(x,y)\in T} x + \max_{(x,y)\in T} y - \min_{(x,y)\in T} y.$$

We denote by STEINER(T) the length of a shortest rectilinear (i.e. edge lengths acc. to ℓ_1) Steiner tree for T. Moreover let MST(T) be the length of a minimum spanning tree in the complete graph on T with edge costs ℓ_1.

Prove that:

(a) $\mathrm{BB}(T) \leq \mathrm{STEINER}(T) \leq \mathrm{MST}(T)$;

(b) $\mathrm{STEINER}(T) \leq \frac{3}{2}\,\mathrm{BB}(T)$ for $|T| \leq 5$;

(c) There is no $\alpha \in \mathbb{R}$ s.t. $\mathrm{STEINER}(T) \leq \alpha\,\mathrm{BB}(T)$ for all finite $\emptyset \neq T \subset \mathbb{R}^2$.

(2 + 3 + 2 points)

Exercise 4.2. Let Y be a Steiner tree for terminal set T with $|T| \geq 2$ in which all leaves are terminals. Prove

$$\sum_{t\in T} \Big(|\delta_Y(t)| - 1\Big) = k - 1$$

where k is the number of full components of Y.

(5 points)

Exercise 4.3. Let (G, c, T) be an instance of the STEINER TREE PROBLEM, G connected, $t \in T$ a terminal and $k \in \mathbb{N}$ with $k \geq 1$.
For each of the following functions $V(G) \times 2^T \to \mathbb{R}_{\geq 0}$ decide whether it defines a feasible lower bound for instances of the RECTILINEAR STEINER TREE PROBLEM and prove your statement.

(a) For two feasible lower bounds lb_a and lb_b, define $\max(\mathrm{lb}_a, \mathrm{lb}_b)$ by

$$\max(\mathrm{lb}_a, \mathrm{lb}_b)(v, I) := \max\Big(\mathrm{lb}_a(v, I), \mathrm{lb}_b(v, I)\Big).$$

(b) Define $\mathrm{lb}_{\mathrm{BB}}(v, I) := \mathrm{BB}(\{v\} \cup I)$.

(c) Define $\mathrm{lb}_{\mathrm{mst}}(v, I) := \frac{\mathrm{mst}(\{v\} \cup I)}{2}$. Here $\mathrm{mst}(\{v\} \cup I)$ denotes the cost of a minimal spanning tree in $(G'[\{v\} \cup I], c')$, where (G', c') is the metric closure of (G, c).

(d) Define $\mathrm{lb}_k(v, I) := \max\Big\{\mathrm{smt}(J) \;\Big|\; t \in J \subseteq I \cup \{v\}, |J| \leq k + 1\Big\}$ if $t \in I$ and $\mathrm{lb}_k(v, I) := 0$ otherwise.

(2 + 2 + 2 + 2 points)

Deadline: May 7th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss24/chipss24_ex.html`

In case of any questions feel free to contact me at schlomberg@or.uni-bonn.de.