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Exercise Set 3

Exercise 3.1. In the following we use the notation

$$S_{B_2}(B_n) := \max\{S_{B_2}(f) : f \in B_n\}$$

for $n \in \mathbb{N}$. Further, for $f \in B_n$, let $f_0 := f_{|x_n=0} \in B_{n-1}$ and $f_1 = f_{|x_n=1} \in B_{n-1}$. One can write f as a *decoding circuit* in the following way:

$$f = (\overline{x}_n \wedge f_0) \vee (x_n \wedge f_1). \tag{1}$$

By applying this representation recursively, it follows that $S_{B_2}(B_n) \leq 2^n - 3$. The goal of this exercise is to improve this bound.

(a) Let $S^*_{B_2}(B_k)$ denote the circuit complexity of computing all functions in B_k (such a circuit has 2^{2^k} outputs). Use (1) to show that

$$S^*_{B_2}(B_k) \in \mathcal{O}(2^{2^k}).$$

(b) Use (a) to show that $S_{B_2}(B_n) \in \mathcal{O}(2^n/n)$.

(2+3 points)

Exercise 3.2. Let $n \in \mathbb{N}$. Construct an AND-prefix graph that has

- $\{\wedge\}$-depth at most $\log n + \log\log n + \mathcal{O}(1)$,
- $\{\wedge\}$-size at most $3n + \mathcal{O}(1)$,
- and a maximum fanout of two.

You may use repeaters to bound the fanout.

(5 points)

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Exercise 3.3. Consider the following recursively defined family of Boolean functions $f_{2n,m} \in B_{2n+m+1}$ $(n, m \in \mathbb{N})$:

$$
\begin{aligned}
&f_{0,0}(x_0) &&= x_0, \\
&f_{0,m}(x_0, \ldots, x_m) &&= x_0 \wedge f^*_{0,m-1}(x_1, \ldots, x_m) && (m \geq 1), \\
&f_{2n,m}(x_0, \ldots, x_{2n+m}) &&= x_0 \wedge f_{2n-2,m}(x_2, \ldots, x_{2n+m}) && (m \geq 0, n \geq 1),
\end{aligned}
$$

where $f^*(x_1, \ldots, x_n) := \neg f(\bar{x}_1, \ldots, \bar{x}_n) \in B_n$ is the dual function of a function $f \in B_n$. Prove the following split equation

$$
\begin{aligned}
f_{2n,2k+m+1}(x_0, x_1, \ldots, x_{2n+2k+m+1}) &= f_{2n,2k}(x_0, \ldots, x_{2n+2k}) \\
&\quad \wedge f^*_{2k,m}(x_{2n+1}, x_{2n+2}, \ldots, x_{2n+2k+m+1}).
\end{aligned}
$$

(5 points)

Exercise 3.4. Let $m \in \mathbb{N}$. Show that a circuit C for $f_{0,m}$ over the basis $\{\wedge, \vee\}$ with depth $D(C) \leq \log_2 m + \log_2 \log_2 m + \mathcal{O}(1)$ and size $S(C) \in \mathcal{O}(m \log m)$ can be computed in time $\mathcal{O}(m \log m)$.

(5 points)

Deadline: April 30th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss24/chipss24_ex.html`

In case of any questions feel free to contact me at schlomberg@or.uni-bonn.de.