Second Example: Unboundedness

Simplex Algorithm: Example II: Unboundedness

$$
\begin{array}{lrcrcrcrcl}
\max & x_1 & & & & & & & & \\
s.t. & x_1 & - & x_2 & + & x_3 & & & = & 1 \\
 & -x_1 & + & x_2 & & & + & x_4 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Initial basis: B={3,4}
Simplex Tableau:

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & - & x_1 & + & x_2 \\
x_4 & = & 2 & + & x_1 & - & x_2 \\
\hline
z & = & & & x_1 & &
\end{array}
$$

Simplex Algorithm: Example II: Unboundedness

$$
\begin{array}{lrcrcrcrcl}
\max & x_1 & & & & & & & & \\
s.t. & x_1 & - & x_2 & + & x_3 & & & = & 1 \\
 & -x_1 & + & x_2 & & & + & x_4 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Initial basis: B={3,4}
Simplex Tableau:

$$
\begin{array}{ccccccc}
x_3 & = & 1 & - & x_1 & + & x_2 \\
x_4 & = & 2 & + & x_1 & - & x_2 \\
\hline
z & = & & & x_1 & &
\end{array}
$$

Recent solution: $x = (0, 0, 1, 2)$.

Simplex Algorithm: Example II: Unboundedness

First Tableau:

$$\begin{array}{rcrcrcr} x_3 & = & 1 & - & x_1 & + & x_2 \\ x_4 & = & 2 & + & x_1 & - & x_2 \\ \hline z & = & & & x_1 & & \end{array}$$

Simplex Algorithm: Example II: Unboundedness

First Tableau:

$$\begin{array}{rcrcrcr} x_3 & = & 1 & - & x_1 & + & x_2 \\ x_4 & = & 2 & + & x_1 & - & x_2 \\ \hline z & = & & & x_1 & & \end{array}$$

Only one candidate: x_1. $x_3 = 1 - x_1 + x_2$ is critical. Replace 3 by 1 in B: $B = \{1, 4\}$.

$x_1 = 1 + x_2 - x_3$.

Second Tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 1 & + & x_2 & - & x_3 \\ x_4 & = & 3 & & & - & x_3 \\ \hline z & = & 1 & + & x_2 & - & x_3 \end{array}$$

Simplex Algorithm: Example II: Unboundedness

First Tableau:

$$\begin{array}{rcrcrcr} x_3 & = & 1 & - & x_1 & + & x_2 \\ x_4 & = & 2 & + & x_1 & - & x_2 \\ \hline z & = & & & x_1 & & \end{array}$$

Only one candidate: x_1. $x_3 = 1 - x_1 + x_2$ is critical. Replace 3 by 1 in B: $B = \{1, 4\}$.

$x_1 = 1 + x_2 - x_3$.

Second Tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 1 & + & x_2 & - & x_3 \\ x_4 & = & 3 & & & - & x_3 \\ \hline z & = & 1 & + & x_2 & - & x_3 \end{array}$$

Recent solution:

$x = (1, 0, 0, 3)$.

Simplex Algorithm: Example II: Unboundedness

Second Tableau:

$$
\begin{array}{rcrcrcr}
x_1 & = & 1 & + & x_2 & - & x_3 \\
x_4 & = & 3 & & & - & x_3 \\
\hline
z & = & 1 & + & x_2 & - & x_3
\end{array}
$$

Only one candidate: x_2.

Simplex Algorithm: Example II: Unboundedness

Second Tableau:

$$
\begin{array}{rcrcrcr}
x_1 & = & 1 & + & x_2 & - & x_3 \\
x_4 & = & 3 & & & - & x_3 \\
\hline
z & = & 1 & + & x_2 & - & x_3
\end{array}
$$

Only one candidate: x_2. No constraint for it!

Simplex Algorithm: Example II: Unboundedness

Second Tableau:

$$
\begin{array}{rcrcrcr}
x_1 & = & 1 & + & x_2 & - & x_3 \\
x_4 & = & 3 & & & - & x_3 \\
\hline
z & = & 1 & + & x_2 & - & x_3
\end{array}
$$

Only one candidate: x_2. No constraint for it!

$\Rightarrow$ The LP is unbounded

Second Example: Degeneracy

Simplex Algorithm: Example III: Degeneracy

$$
\begin{array}{lrcrcrcrcl}
\max & & & x_2 & & & & & & \\
s.t. & -x_1 & + & x_2 & + & x_3 & & & = & 0 \\
 & x_1 & & & & & + & x_4 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Initial basis: $B = \{3, 4\}$

Simplex Tableau:

$$
\begin{array}{rcrcrcr}
x_3 & = & & & x_1 & - & x_2 \\
x_4 & = & 2 & - & x_1 & & \\
\hline
z & = & & & & & x_2
\end{array}
$$

Simplex Algorithm: Example III: Degeneracy

$$
\begin{array}{rrcrcrcrcl}
\max & & & x_2 & & & & & & \\
s.t. & -x_1 & + & x_2 & + & x_3 & & & = & 0 \\
 & x_1 & & & & & + & x_4 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Initial basis: $B = \{3, 4\}$

Simplex Tableau:

$$
\begin{array}{rcrcrcr}
x_3 & = & & & x_1 & - & x_2 \\
x_4 & = & 2 & - & x_1 & & \\
\hline
z & = & & & & & x_2
\end{array}
$$

$\Rightarrow x = (0, 0, 0, 2)$: degenerated solution.

Simplex Algorithm: Example III: Degeneracy

First Tableau:

$$\begin{array}{rcrcrcr} x_3 & = & & & x_1 & - & x_2 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & & & & & x_2 \end{array}$$

Want to increase x_2. $x_3 = x_1 - x_2$ is critical. Replace 3 by 2 in B:
$B = \{2, 4\}$.
$x_2 = x_1 - x_3$. We will replace 3 by 2 in the basis.
But: We cannot increase x_2.
Second Tableau:

$$\begin{array}{rcrcrcr} x_2 & = & & & x_1 & - & x_3 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & & & x_1 & - & x_3 \end{array}$$

Recent solution: $x = (0, 0, 0, 2)$.

Simplex Algorithm: Example III: Degeneracy

Second Tableau:

$$\begin{array}{rcrcrcr} x_2 & = & & & x_1 & - & x_3 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & & & x_1 & - & x_3 \end{array}$$

Increase x_1. $x_4 = 2 - x_1$ is critical. $x_1 = 2 - x_4$. New base $B = \{1, 2, 0, 0\}$.

Third Tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 2 & & & - & x_4 \\ x_2 & = & 2 & - & x_3 & - & x_4 \\ \hline z & = & 2 & - & x_3 & - & x_4 \end{array}$$

Optimum solution: $x = (2, 2, 0, 0)$.

The Simplex Algorithm

Algorithm 1: Simplex Algorithm

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$

Output: $\tilde{x} \in \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ maximizing $c^t x$ or the message that $\max\{c^t x \mid Ax = b, x \geq 0\}$ is unbounded or infeasible

1 Compute a feasible basis B;

2 If no such basis exists, stop with the message “INFEASIBLE”;

3 Set $N = \{1, \ldots, n\} \setminus B$ and compute the feasible basic solution x for B;

4 Compute the simplex tableau $\begin{array}{ccccc} x_B & = & p & + & Qx_N \\ \hline z & = & z_0 & + & r^t x_N \end{array}$ for B;

5 **if** $r \leq 0$ **then**

  return $\tilde{x} = x$;

6 Choose $\alpha \in N$ with $r_\alpha > 0$;

7 **if** $q_{i\alpha} \geq 0$ **for all** $i \in B$ **then**

  return “UNBOUNDED”;

8 Choose $\beta \in B$ with $q_{\beta\alpha} < 0$ and $\frac{p_\beta}{q_{\beta\alpha}} = \max\{\frac{p_i}{q_{i\alpha}} \mid q_{i\alpha} < 0, i \in B\}$;

9 Set $B = (B \setminus \{\beta\}) \cup \{\alpha\}$;

10 GOTO line 3;