Research Institute for Discrete Mathematics
Chip Design
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Exercise Set 7

Exercise 7.1:
Recall the LP for the d-dimensional arrangement problem from the lecture for $d = 2$:

$$\begin{array}{llrcll}
\min & & \displaystyle\sum_{e \in E(G)} w(e) l(e) & & & \\
\text{s.t.} & & \displaystyle\sum_{y \in X} l(\{x,y\}) & \geq & \frac{1}{4}(|X|-1)^{1+1/2} & \forall\, X \subseteq V(G),\ \forall\, x \in X \\
& & l(\{x,y\}) + l(\{y,z\}) & \geq & l(\{x,z\}) & \forall\, x,y,z \in V(G) \\
& & l(\{x,y\}) & \geq & 0 & \forall\, x,y \in V(G) \\
& & l(\{x,x\}) & = & 0 & \forall\, x \in V(G) \\
& & l(e) & \geq & 1 & \forall\, e \in E(G)
\end{array}$$

Let L be the optimal value of this LP. Show that there is no feasible solution of the given instance of the 2-dimensional arrangement problem with cost less than L.

(4 points)

Exercise 7.2:
Let $G = (V, E)$ be an undirected graph with edge weights $w : E(G) \to \mathbb{R}_{\geq 0}$. Let $\mathcal{C} \subset V(G)$ and $x : V(G) \setminus \mathcal{C} \to \{1, \ldots, k\}$ be a placement function, where $k \in \mathbb{N}$. We are looking for positions $x : \mathcal{C} \to \{1, \ldots, k\}$ such that

$$\sum_{e=\{v,w\} \in E(G)} w(e) \cdot |x(v) - x(w)|$$

is minimum. It is allowed to place several vertices at the same position.

Prove that this problem can be solved optimally by solving $k-1$ minimum weight s-t cut problems in digraphs with $\mathcal{O}(|V(G)|)$ vertices and $\mathcal{O}(|E(G)|)$ edges.

Hint: Consider the digraphs G_j defined by $V(G_j) = \{s,t\} \cup \mathcal{C}$ and

$$\begin{aligned}
E(G_j) = &\{\{s,v\} : \exists\, w \in V(G) \setminus \mathcal{C}, x(w) \leq j, \{v,w\} \in E(G)\} \cup \\
&\{\{v,w\} : v,w \in \mathcal{C}, \{v,w\} \in E(G)\} \cup \\
&\{\{v,t\} : \exists\, w \in V(G) \setminus \mathcal{C}, x(w) > j, \{v,w\} \in E(G)\}
\end{aligned}$$

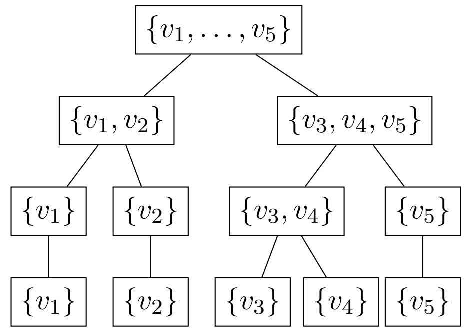
(6 points)

Definition:

Let $G = (V, E)$ be an undirected graph. A $\frac{3}{4}$*-balanced hierarchical decomposition* P of G is a sequence $P_0, P_1, \ldots, P_m$ such that

- $P_0 = \{V(G)\}$
- P_{i+1} is a refinement of P_i for each $i = 0, 1, \ldots, m-1$
- $P_m = \{\{v\} : v \in V(G)\}$
- $|W| \leq \left(\frac{3}{4}\right)^i \cdot |V(G)|$ for all $W \in P_i$

For an edge edge $e = \{v, w\} \in E(G)$ we denote by $l(e)$ the index i such that v and w belong to the same set in P_i but not in P_{i+1}.



Example of a $\frac{3}{4}$-balanced hierarchical decomposition. Sets in the ith row belong to P_{i-1}. In the picture, $l(\{v_1, v_2\}) = 1$ and $l(\{v_3, v_4\}) = 2$.

Exercise 7.3:
Consider the following two problems, where $d \in \mathbb{N}$ is a constant:

Input: An undirected graph $G = (V, E)$.

Output of the min. cost $\frac{3}{4}$-balanced hier. dec. problem:
A $\frac{3}{4}$-balanced hier. dec. of G minimizing $\sum_{e \in E(G)} \sqrt[d]{\left(\frac{3}{4}\right)^{l(e)} \cdot |V(G)|}$.

Output of the min. cost linear arr. problem with d-dim. costs:
A linear arrangement p of G minimizing $\sum_{e=\{v,w\} \in E(G)} \sqrt[d]{|p(v) - p(w)|}$.

Prove that an approximation algorithm with approximation ratio α for the *minimum cost $\frac{3}{4}$-balanced hierarchical decomposition problem* yields an approximation algorithm with approximation ratio $\mathcal{O}(\alpha)$ for the *minimum cost linear arrangement problem with d-dimensional costs.*

(6 points)

Deadline: Thursday, June 20th, before the lecture.