Research Institute for Discrete Mathematics
Chip Design
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Exercise Set 4

Exercise 4.1:
A Steiner topology is called *full* if all terminals have degree one. Let $f(k)$ denote the number of full topologies for the Rectilinear Steiner tree problem with k terminals in which all Steiner points have degree 3. Derive and prove a formula on $f(k)$.

(4 points)

Exercise 4.2:
Let F be a shortest rectilinear Steiner tree for a set $Z \subset \mathbb{R}^2$ of terminals. For $u, v \in V(F)$ define

$$\mathcal{L}(u,v) := \{p \in \mathbb{R}^2 : ||p-u||_1 < ||u-v||_1 \text{ and } ||p-v||_1 < ||u-v||_1\}$$
$$\mathcal{R}(u,v) := \{p \in \mathbb{R}^2 : ||u-v||_1 = ||u-p||_1 + ||p-v||_1\}$$

Prove:

(a) If $\{u,v\} \in E(F)$, then $\mathcal{L}(u,v)$ contains no terminal, Steiner point or interior segment point of F.

(b) If $u, v, w \in E(F)$ such that $\{u,w\}$ and $\{w,v\}$ are perpendicular segments of F, then $\mathcal{R}(u,v)^\circ$ contains no terminal, Steiner point or interior segment point of F.

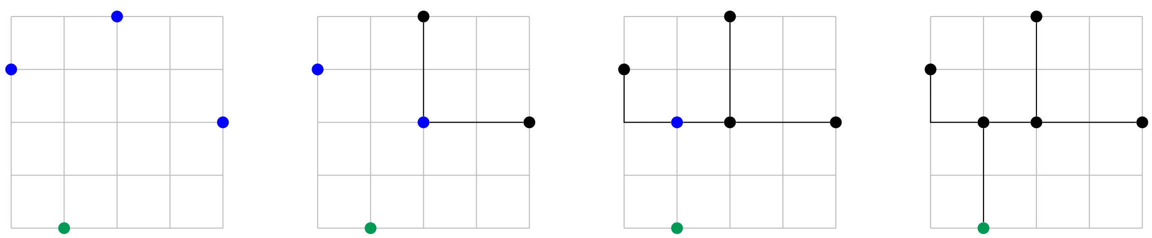
(2+2 points)

Definition: Let $T \subset \mathbb{R}^2$ be a finite set and let $r \in \mathbb{R}^2$.

- A *rectilinear shortest path tree* for $T+r$ is a rectilinear Steiner tree F for instance $T \cup \{r\}$ such that the r-t path contained in F is a shortest path w.r.t. L_1 distances for all $t \in T$.
- A *minimum cost rectilinear shortest path tree* for $T+r$ is a rectilinear shortest path tree F for which $c(F) := \sum_{\{v,w\} \in E(F)} ||v-w||_1$ is minimum.
- We denote the cost of a minimum cost rectilinear shortest path tree by $\text{rspt}(T+r)$.
- For $p_1, p_2, p_3 \in \mathbb{R}^2$ we define $\text{med}(p_1,p_2,p_3) \in \mathbb{R}^2$ to be the point with x- (resp. y) coordinate equal to the median of the x- (resp. y) coordinates of p_1, p_2 and p_3.

Exercise 4.3:
Consider the following algorithm:

```
F := (T ∪ {r}, ∅)
while |T| > 1 do
    choose t_1, t_2 ∈ T maximizing ||med(r, t_1, t_2) − r||_1
    include med(r, t_1, t_2) to V(F)
    connect med(r, t_1, t_2) with t_1 and t_2 in F
    set T = (T\{t_1, t_2}) ∪ {med(r, t_1, t_2)}
include an edge between r and the remaining element of T to E(F)
return F
```



Example of the algorithm. Root r is plotted in green. The blue vertices depict T.

Let F be the output of the algorithm. Prove:

(a) F is a rectilinear shortest path Steiner tree.

(b) $c(F) \leq 2 \cdot \text{rspt}(T + r)$ if $r = (0, 0)$ and all terminals $t \in T$ are located in the first quadrant. *(Remark: The result holds for general instances)*

(c) The approximation ratio of 2 is tight.

(d) The algorithm can be implemented to run in $\mathcal{O}(|T| \cdot \log(|T|))$ time.

(2+6+4+4 points)

Deadline: Thursday, May **16**th, before the lecture.