Bonn Problem-Solving Seminar, BPS 3. August 2, 2013

András Frank
frank@cs.elte.hu

(In the solutions, it is feasible to rely on such basic results as Kőnig theorem, Menger theorem, MFMC theorem and algorithm, Dijkstra algorithm, Hoffman's theorem on feasible circulations, etc.)

1. Prove that a k-edge-connected graph $G = (V, E)$ contains k spanning trees the union of which is k-edge-connected.

2. Prove that the edge-set E of a simple connected graph can be partitioned into paths of length two provided that $|E|$ is even.

3. Prove that the edge-set of a graph can be coloured by red and blue do that the $|d_{red}(v) - d_{blue}(v)| \leq 2$ for every node v.

4. Suppose that a graph $G = (V, E)$ includes k edge-disjoint spanning trees. Prove that, given a specified subset $F \subset E$ of k elements, there are k edge-disjoint spanning trees in such a way that each of them contains at most one element of F.

5. Given a connected graph $G = (V, E)$ with a weight-function w on E. Develop a polynomial algorithm that finds a subset of edges of minimum cardinality that intersects every spanning tree of maximum weight.

6. Suppose that some of the edges of a digraph $D = (V, A)$ are red. If there is a spanning arborescence of root r containing k red edges and there is a spanning arborescence of root r containing $k + 2$ red edges, then there is a spanning arborescence of root r containing $k + 1$ red edges.

7. We call a sequence $a_1, a_2, \ldots, a_k$ of numbers **convex** if $a_1 - a_2 \geq a_2 - a_3 \geq \cdots \geq a_{k-1} - a_k$. Develop a polynomial algorithm that computes the largest convex subsequence of an input sequence $b_1, \ldots, b_n$.

8. Let $H = (V, \mathcal{E})$ by a hypergraph in which every hyperedge has at least k elements and the degree of every node is at most $k - 1$. Prove that the nodes can be coloured by red and blue in such a way that no unicoloured hyperedge exists.

9. Prove that every strongly connected digraph can be made Eulerian by adding new edges in parallel with existing ones.

10. We are given n di-circuits in a digraph on n nodes. Prove that it is possible to select one edge from each of the n di-circuits so that the set of selected edges contains a di-circuit.