Bonn Problem-Solving Seminar, BPS 1. July 19, 2013

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(In the solutions, it is feasible to rely on such basic results as Kőnig theorem, Menger theorem, MFMC theorem and algorithm, Dijkstra algorithm, etc.)

1. Given two posets P_1 and P_2 on a common groundset V, prove that there is a subset $A \subseteq V$ which is an antichain both in P_1 and in P_2 such that, for every element $x \in V - A$, there is a $p \in A$ that is larger than x in at least one of the two posets.

2. Let $D = (V, A)$ be a digraph with two specified nodes s and t. Design a polynomial algorithm to find two disjont subsets S and T of V for which $s \in S$, $t \in T$ and $\delta(S) + \delta(T)$ is as small as possible where $\delta(X)$ denotes the number of edges leaving X.

3. Prove that a 2-edge-connected graph has a smooth strongly connected orientation. (Smooth means that $|\varrho(v) - \delta(v)| \leq 1$ for every node $v \in V$.)

4. Design a polynomial algorithm to decide for a bipartite graph $G = (S, T; E)$ and positive integer k whether

(A) $|\Gamma(X)| \geq |X| + 1$ holds for every nonempty $X \subseteq S$,

(B) $|\Gamma(X)| \geq |X| + k$ holds for every nonempty $X \subseteq S$.

5. An interval I is the union of the set $\mathcal{I} = \{I_1, \dots, I_k\}$ of closed subintervals. Prove that it is possible to select some pairwise disjoint members of $\mathcal{I}$ so that their total length is at least half of the length of I.

6. Decide if the following statement is true or not. If a poset can be partitioned into longest chains, then it can be partitioned into largest antichains.

7. Let D be an acycilic digraph and $k \geq 2$ an integer. Design a polynomial time algorithm for deciding whether or not every circuit C of D has at least $|C|/k$ edges in both directions.

8. Let $G = (V, E)$ be a k-edge-connected graph with $|V| \geq 2$ that is minimal in the sense that $G - e$ is not k-edge-connected for every $e \in E$. Prove that G has a node of degree k. Is it true that G always has two such nodes?

9. We placed the nodes of the two colour classes of an edge-weighted bipartite graph $G = (S, T; E)$ on two horizontal lines in the plane. The edges of G are represented by straight line segments. Two such edges are said to be crossing if they share an inner node in common.

(A) Design a polynomial algorithm to compute a cross-free matching in G whose total weight is maximum.

(B) Design a polynomial algorithm to compute a cross-free forest in G whose total weight is maximum.

10. Prove that a tournament includes a node from which every other node can be reached by a one-way path of length at most 2.