d-dimensional arrangement revisited

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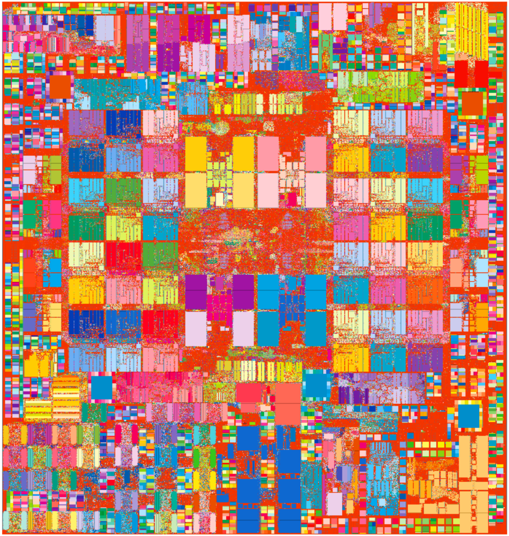
(joint work with Daniel Rotter)

April 16, 2013

Chip design



Chip design

Placement Problem:

Given

a (large) set of rectangular objects with pins,

a rectangular chip area,

fixed objects and/or pins,

and a partition of the set of pins into nets;

place the objects within the chip area without overlaps such that

“the pins of every net are close to each other”

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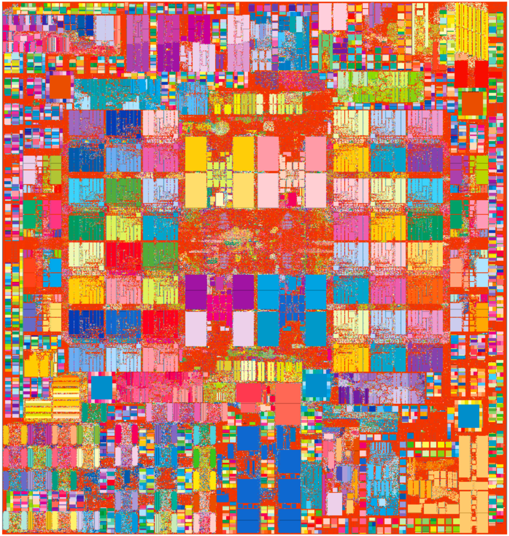

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Simplifications: all objects are unit squares with pins in the center, each net has two pins, no fixed objects/pins, measure total ℓ_1-length

Models with polylogarithmic approximation algorithms

- Vempala [1998]: minimize total length and maximum edge length, $O(\log^{3.5} n)$-approximation
- Even, Guha, Schieber [2000]: embed edges by edge-disjoint paths, minimize area, $O(\log^4 n)$-approximation

where n is the number of objects that are to be placed.

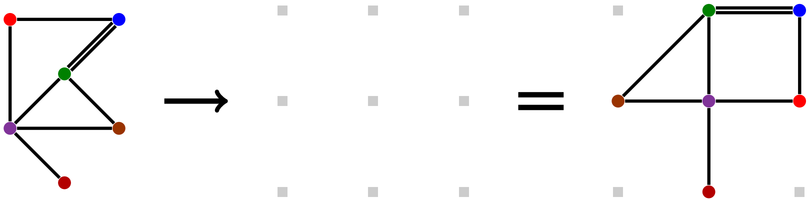
In the following:

- 2-dimensional arrangement: minimize total length only

d-dimensional arrangement problem

Given: undirected graph $G = (V, E)$ and $k \geq \sqrt[d]{|V|}$

Find: injection $p\colon V \to \{1, \ldots, k\}^d$ minimizing $\sum_{\{v,w\} \in E} ||p(v) - p(w)||_1$



- $d = 1$: linear arrangement problem
- $d = 2$: interesting model of placement in chip design
- this talk: $d \geq 2$ fixed; unit weights, but all results generalize to weighted version (given edge weights, take weighted sum)

Approximation algorithms for d-dimensional arrangement

Given: undirected graph $G = (V, E)$ and $k \geq \sqrt[d]{|V|}$

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- $d = 1$: linear arrangement problem
  - *NP*-hard: Garey, Johnson [1976]
  - $O(\sqrt{\log n} \log \log n)$-approximation: Feige, Lee [2007] and independently Charikar, Hajiaghayi, Karloff, Rao [2010], improving on Rao, Richa [2004]

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- $d \geq 2$ (fixed):
  - sketch of $O(\log^2 n)$-approximation by recursive bipartitioning: Hansen [1989], using Leighton, Rao [1999]
  - can lead to $O(\log^{3/2} n)$-approximation when using Arora, Rao, Vazirani [2009]

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Reduction to linear arrangement with "d-dimensional cost"

$$\min \left\{ \sum_{\{v,w\} \in E} \sqrt[d]{|p(v) - p(w)|} \;\middle|\; p : V \to \{1, \ldots, n\} \text{ bijective} \right\}$$

- reduction proposed by Even, Naor, Rao, Schieber [2000]
- $O(\log n \log\log n)$-approximation for this problem by Even et al. [2000]
- $O(\sqrt{\log n})$-approximation for this problem by Charikar, Makarychev, Makarychev [2007]

Reduction to linear arrangement with “*d*-dimensional cost”

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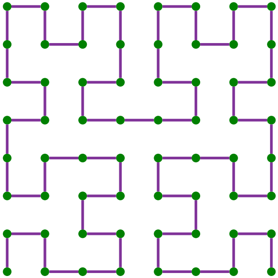
Unfortunately, this does not imply the same approximation ratios for *d*-dimensional arrangement!

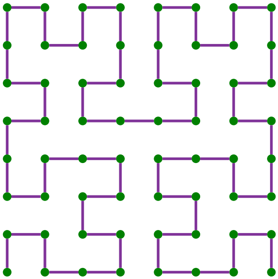
Reduction to linear arrangement with d-dimensional cost

Lemma

Using Hilbert's space-filling curve, we can find an injection $p : \{1, \ldots, n\} \to \{1, \ldots, k\}^d$ *such that* $||p(i) - p(j)||_1 \leq 4(d+1)\sqrt[d]{|i-j|}$ *for all* i, j.



Reduction to linear arrangement with *d*-dimensional cost



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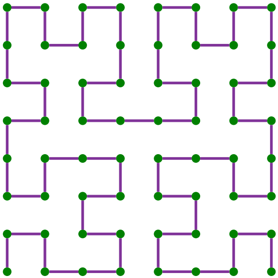
Corollary

Given a linear arrangement of G with d-dimensional cost γ, *we can compute a d-dimensional arrangement of G with cost* $O(\gamma)$.

(Even, Naor, Rao, Schieber [2000])

Reduction to linear arrangement with *d*-dimensional cost



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However,
the optimum *d*-dimensional cost for a linear arrangement can be much larger than the optimum cost of a *d*-dimensional arrangement.

How good is the reduction?

Theorem

For any graph $G = (V, E)$ and any injection $p : V \to \{1, \ldots, k\}^d$, there exists a bijection $q : V \to \{1, \ldots, n\}$ such that

$$\sum_{\{v,w\} \in E} \sqrt[d]{|q(v) - q(w)|} \;\leq\; O(\log n) \sum_{\{v,w\} \in E} ||p(v) - p(w)||_1 \,.$$

There are pairs (G, p) for which this bound is tight.

Therefore a factor $O(\log n)$ is lost in this reduction!

Upper bound

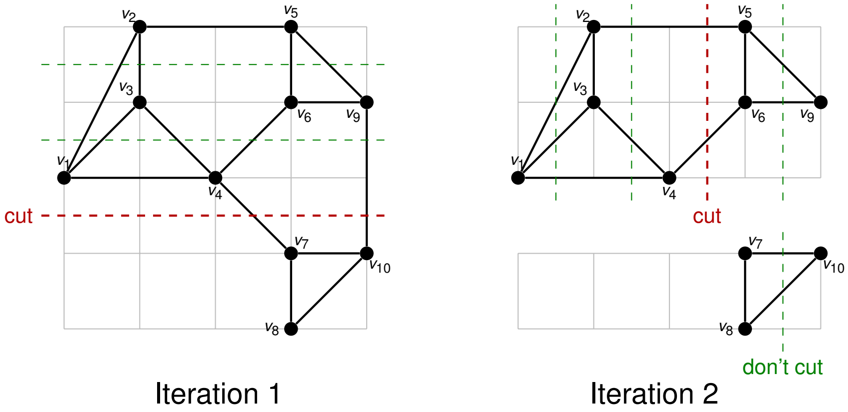
Lemma

For any graph $G = (V, E)$ and any injection $p : V \to \{1, \ldots, k\}^d$ (where $k, d \in \mathbb{N}$), there exists a bijection $q : V \to \{1, \ldots, n\}$ such that

$$\sum_{\{v,w\} \in E} \sqrt[d]{|q(v) - q(w)|} \;\leq\; 32 d \ln n \sum_{\{v,w\} \in E} ||p(v) - p(w)||_1 \,.$$

Proof of upper bound (sketch)

- Consider cut coordinates with enough vertices on both sides
- If sufficiently many such coordinates exist, take smallest cut
- Continue with next dimension until each set is a singleton



Yields balanced hierarchical decomposition; order vertices accordingly

Lower bound

Consider the d-dimensional hypercube graph (V_k^d, E_k^d):

- $V_k^d = \{1, \ldots, k\}^d$
- $E_k^d = \big\{\{x, y\} : x, y \in V_k^d,\ ||x - y||_1 = 1\big\}$

The identity function is a d-dimensional arrangement of cost $\sum_{\{v,w\}\in E_k^d} ||v - w||_1 = |E_k^d| = d(k^d - k^{d-1}) < dn$.

Lemma

Let $d \geq 2$. If $q : V_k^d \to \{1, \ldots, n\}$ is any bijection, then

$$\sum_{\{v,w\}\in E_k^d} \sqrt[d]{|q(v) - q(w)|} >$$

$$\tfrac{3}{16}\left(1 - \left(\tfrac{3}{4}\right)^{\frac{d-1}{d}}\right)\left(1 - \left(\tfrac{3}{4}\right)^{1/d}\right) d\, n \log_2 n - \tfrac{3dn}{64}.$$

Spreading LP

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{e=\{v,w\}\in E} l(v,w) & & & \\
\text{s.t.} & l(v,w) = l(w,v) & \geq & 0 & (v,w \in V) \\
& l(u,v) + l(v,w) & \geq & l(u,w) & (u,v,w \in V) \\
& \displaystyle\sum_{u\in U} l(u,v) & \geq & \dfrac{(|U|-1)^{1+1/d}}{4} & (U \subseteq V,\ v \in U)
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- Can be solved in polynomial time (Even, Naor, Rao, Schieber [2000], Bornstein, Vempala [2004])

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- Can be solved in polynomial time (Even, Naor, Rao, Schieber [2000], Bornstein, Vempala [2004])
- LP value is a lower bound on the optimum cost of a d-dimensional arrangement (Even [2011])
- Implies that the Even-Naor-Rao-Schieber algorithm is indeed an $O(\log n \log \log n)$-approximation algorithm

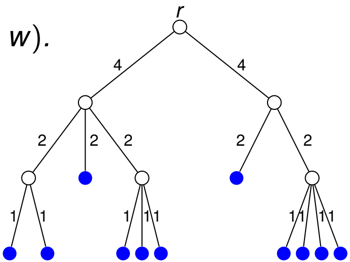
Approximating any metric by a tree metric

Lemma (Fakcharoenphol, Rao and Talwar [2004])

Let $G = (V, E)$ be a graph, $n = |V| \geq 2$, and $l : V \times V$ a metric. Then one can compute in polynomial time a 2-hierarchically well-separated tree (T, r, c) such that V is the set of leaves of T, and the induced tree metric l' satisfies:

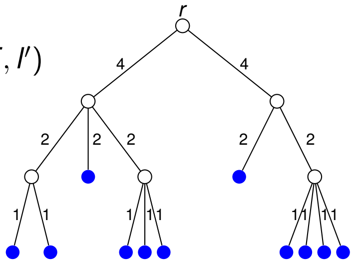
(a) $l'(v, w) \geq l(v, w)$ *for all* $v, w \in V$, *and*

(b) $\displaystyle\sum_{\{v,w\} \in E} l'(v, w) \leq O(\log n) \sum_{\{v,w\} \in E} l(v, w)$.



$O(\log n)$-approximation algorithm for d-dimensional arrangement

- Solve spreading LP $\longrightarrow l$
- Approximate l by tree metric $\longrightarrow (T, l')$
- Order of the leaves of T in the natural way $\longrightarrow q : V \to \{1, \ldots, n\}$
- Arrange the vertices according to the Hilbert curve lemma.



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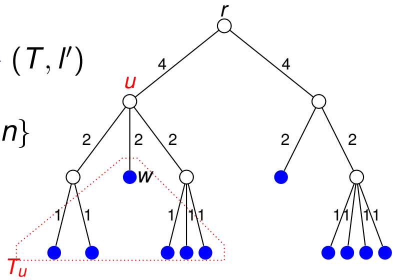

Proof of approximation ratio:

- Let $\{v, w\} \in E$ and u the nearest common ancestor of v and w.

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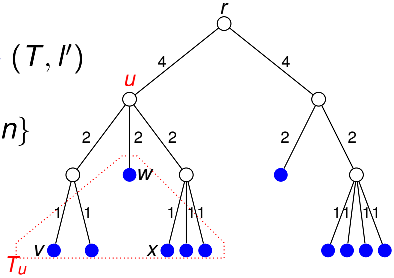

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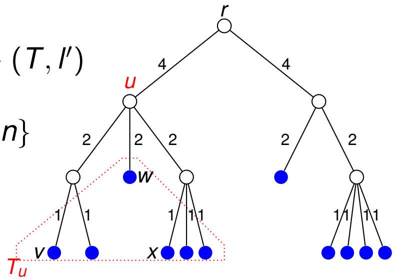

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$$\sqrt[d]{|q(v) - q(w)|} \leq \sqrt[d]{|T_u| - 1} \leq 4\, l(v, x) \leq 4\, l'(v, x) \leq 8\, l'(v, w).$$

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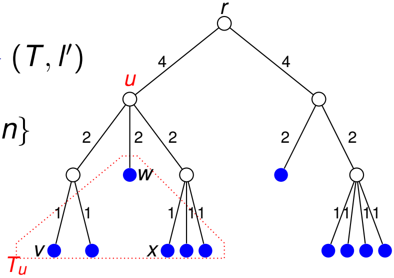

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- Hence,

$$\sum_{\{v,w\} \in E} \sqrt[d]{|q(v) - q(w)|} \leq 8 \sum_{\{v,w\} \in E} l'(v, w) \leq O(\log n) \sum_{\{v,w\} \in E} l(v, w).$$

$\square$

Discussion

Currently best approximation algorithm (see above)

- finds a "high-dimensional embedding" (spreading metric),
- approximates it by a tree metric (by recursive bipartitioning),
- makes it a linear order,
- makes it d-dimensional via a space-filling curve.

Poly-time but very slow. No fixed pins etc. $O(\log n)$-approximation

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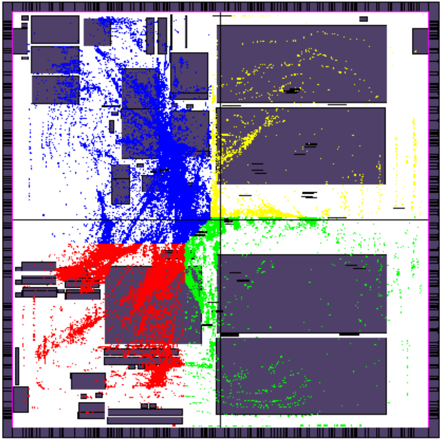
Some of the best heuristics in practice ($d = 2$) proceed as follows (e.g., BonnPlace, Brenner, Struzyna, Vygen [2008]):

- finds a 2-dimensional embedding (quadratic placement)
- uses recursive quadrisection (Vygen [2005])
- concludes with legalization (Brenner, Vygen [2004])

Fast. Needs fixed pins for “spreading”. No approximation guarantee

Open problems

Bring
theory and practice
closer together



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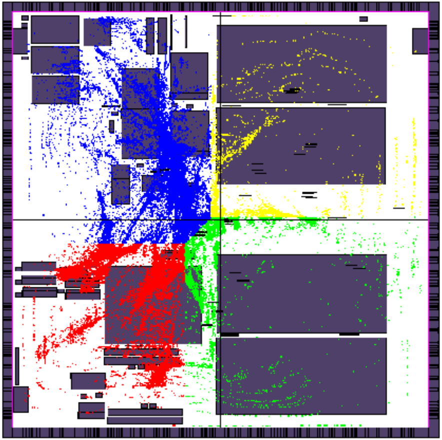

Bring
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closer together

- Improve approximation guarantee (or prove hardness)
- Obtain $O(\log n)$-approximation without spreading LP
- Generalize to practically more relevant problems
- Prove approximation guarantee for a practical algorithm