A $\left(\frac{3}{2} + \frac{1}{e}\right)$-Approximation Algorithm for Ordered TSP

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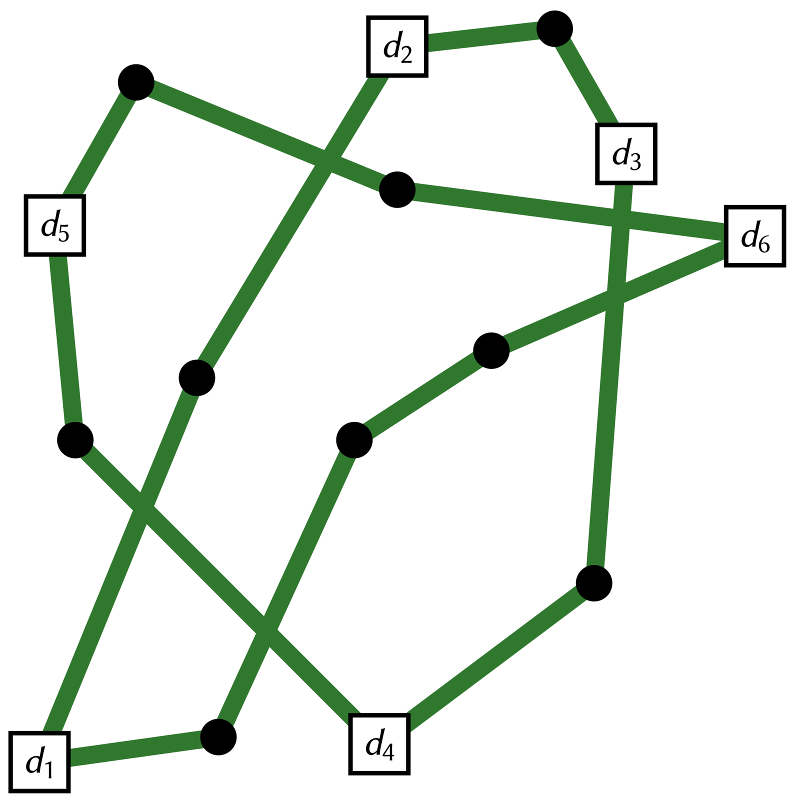
Example instance



Optimal solution wrt. the underlying Euclidean metric

The Ordered Travelling Salesperson Problem

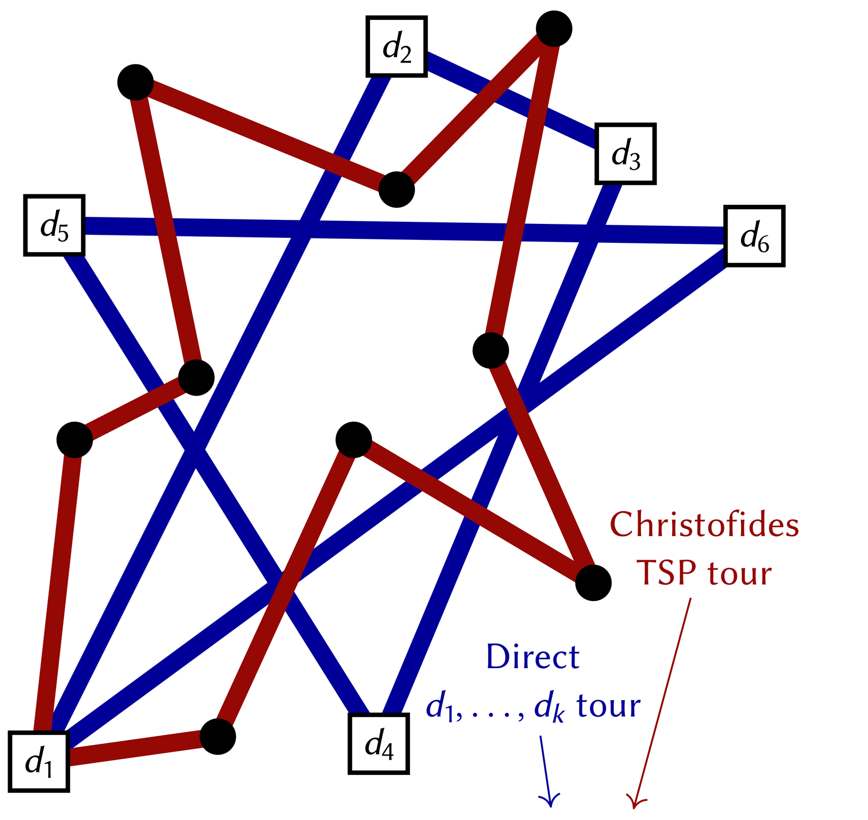
Input: Complete graph $G = (V, E)$, metric edge costs $c\colon E \to \mathbb{R}_{\geq 0}$, distinct vertices $d_1, \dots, d_k \in V$.

Task: Find a cheapest Hamiltonian cycle C in G that visits $d_1, \dots, d_k$ in this order.

Main result

There is a polynomial-time α-approximation algorithm for Ordered TSP, where $\alpha = \frac{3}{2} + \frac{1}{e} < 1.868$.

Previous results



Combined tour: $\alpha = 1 + \frac{3}{2} = \frac{5}{2}$

Böckenhauer, Mömke, Steinova: $\alpha = \frac{5}{2} - \frac{2}{k}$

[J. Discr. Alg., 2013]

Our Algorithm

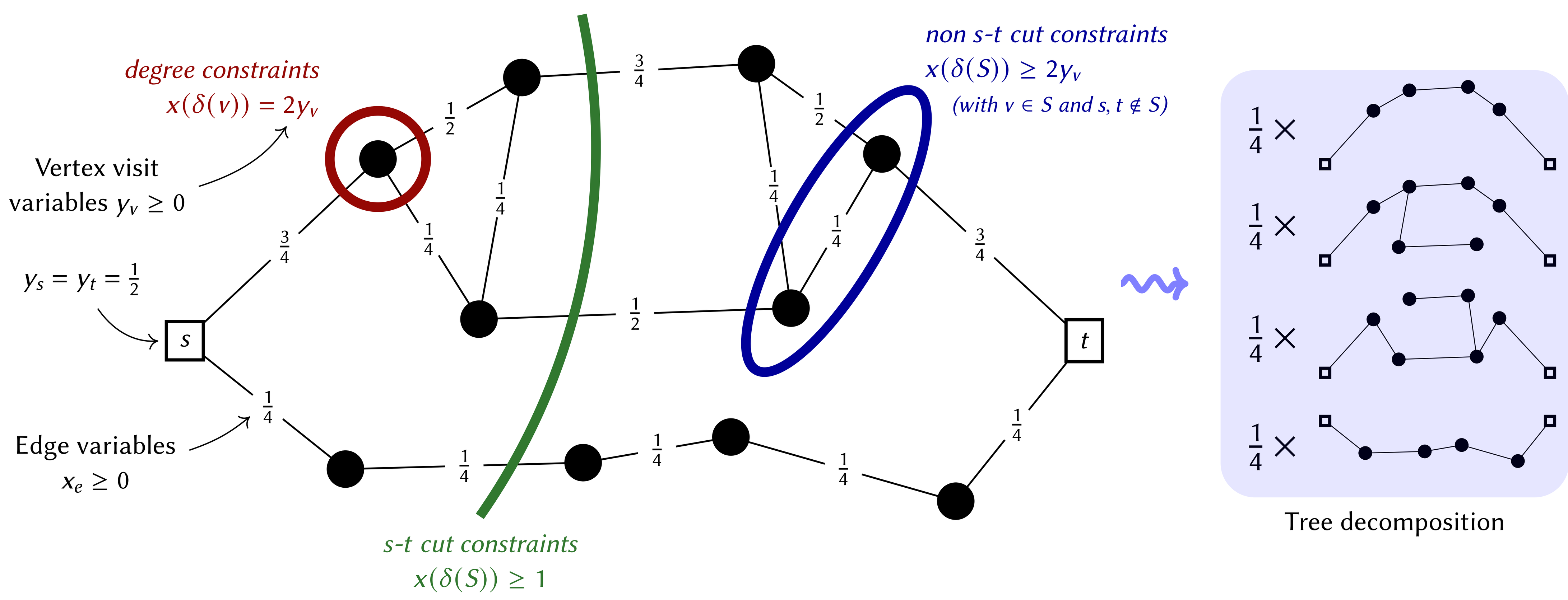
— a randomized LP rounding approach —

1 Solve the LP relaxation

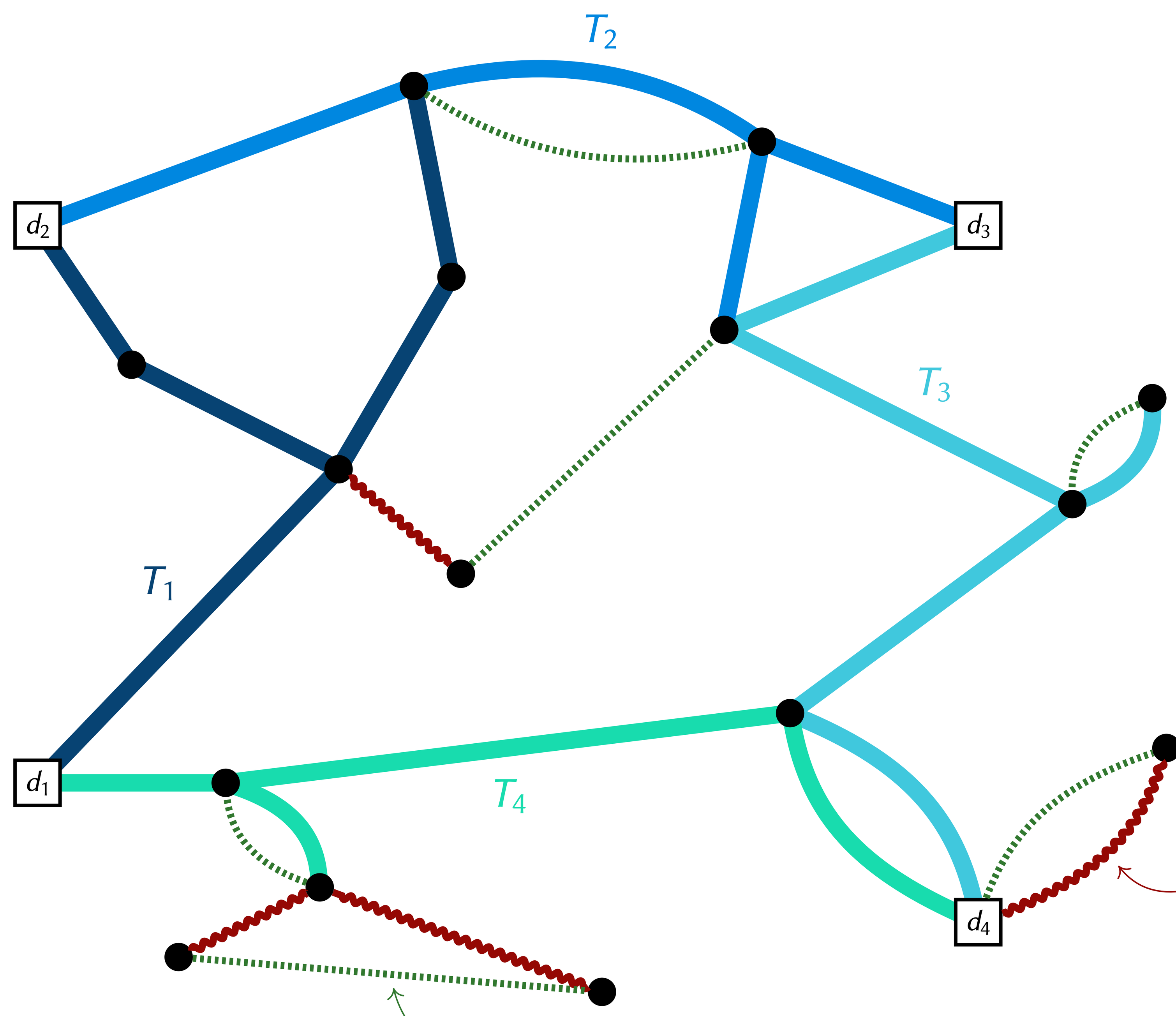
$$c_{LP} := \min \sum_{e \in E} c_e \sum_{i=1}^{k} x_e^i$$
$$\sum_{i=1}^{k} y_v^i = 1 \qquad \forall v \in V$$
$$(x^i, y^i) \in P_{d_i\text{-}d_{i+1}} \; \forall i \in \{1, \dots k\}$$

Here, $P_{s\text{-}t}$ is a Held-Karp type relaxation of s-t paths.

A point $(x, y) \in P_{s\text{-}t}$ and its decomposition



Tree decomposition



2 Sample trees from LP solution

Decompose (x^i, y^i) and sample a tree T_i connecting d_i and d_{i+1} with

$$\mathbb{E}[c(E[T_i])] = c^\top x^i \quad \text{and}$$
$$\mathbb{P}[v \in V[T_i]] = y_v^i \quad \text{for } v \in V \setminus \{d_i, d_{i+1}\} \; .$$

3 Ensure connectivity

Vertex $v \neq d_j$ is not covered by *any* T_i with probability

$$\prod_{i=1}^{k} (1 - y_v^i) \leq \exp\left(-\sum_{i=1}^{k} y_v^i\right) = \frac{1}{e} \; .$$

$\Longrightarrow$ can find a connector at expected cost $\leq \frac{1}{e} \cdot c_{LP}$.

4 Parity correction

Obtain even degrees through a suitable T-join at cost $\leq \frac{1}{2} \cdot c_{LP}$.

5 Tour construction

Shortcut an Euler tour respecting the order $d_1, \dots, d_k$.

Derandomization: Straightforward using the *method of conditional expectations*.

Generalization: Polynomial-time $(\ell + 1/2 + 1/e^\ell)$-approximation for *Precedence-Constrained TSP* with ℓ independent linear orders.

Full paper: arXiv:2405.06244

DFG Deutsche Forschungsgemeinschaft

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