Approximation Algorithms for Traveling Salesmen

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University of Bonn

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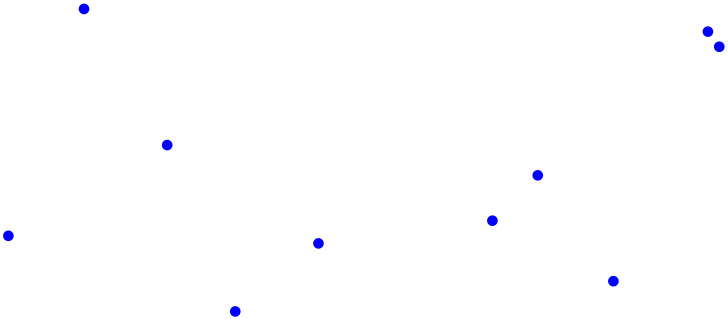

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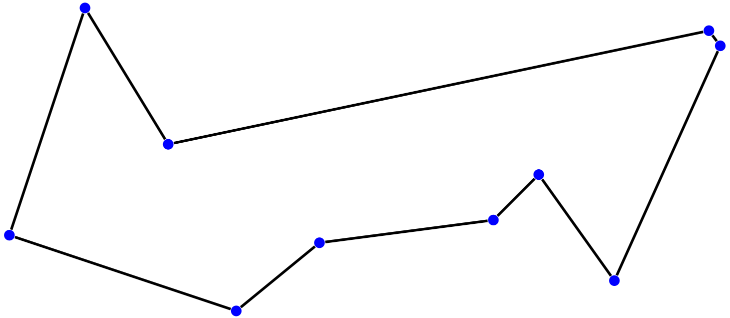

Variants of the TSP

- start = end?
- symmetric or asymmetric?
- triangle inequality?
- visit every city at least once or exactly once?

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Whether we assume triangle inequality or allow visiting cities more than once is equivalent.

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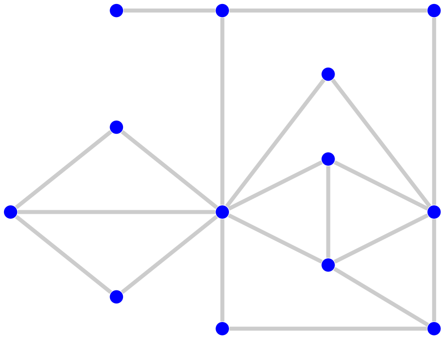
Given a connected graph G, find a minimum length closed edge progression in G that visits every vertex at least once.



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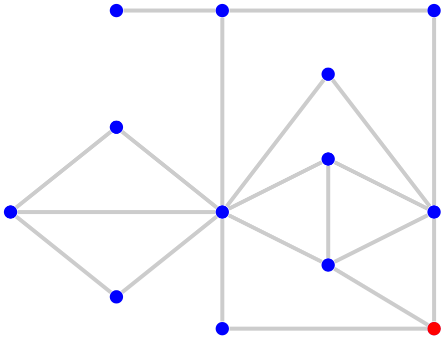

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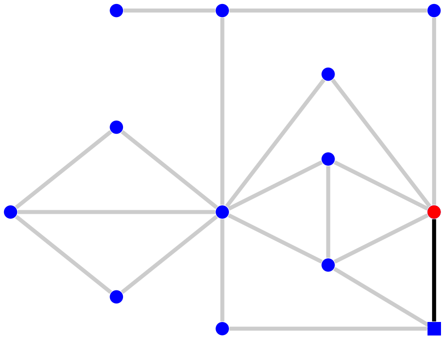

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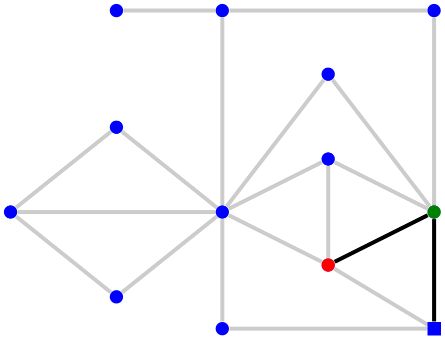

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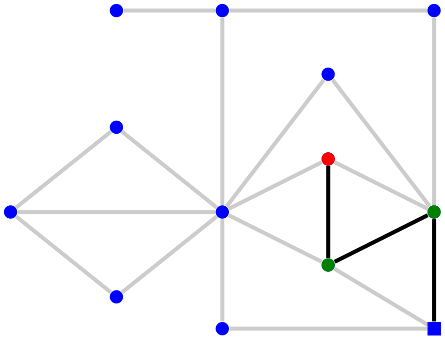

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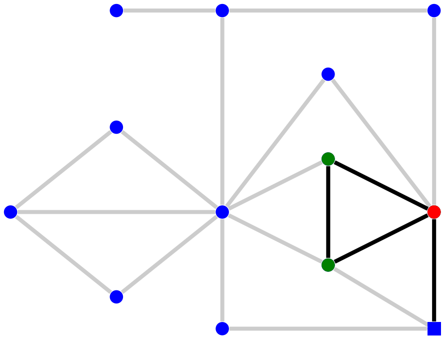

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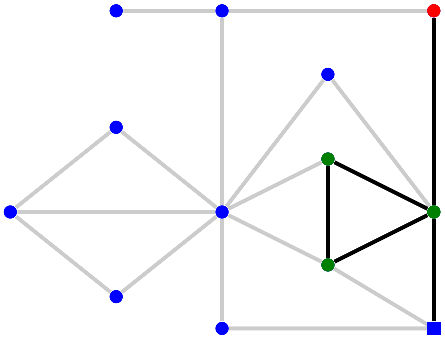

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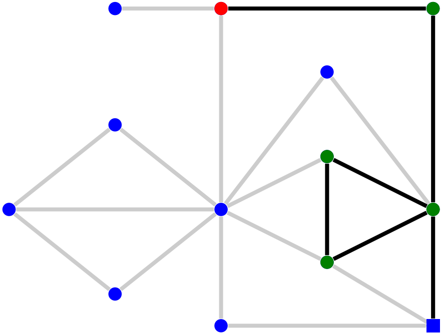

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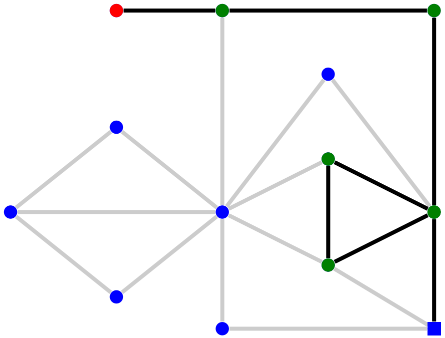

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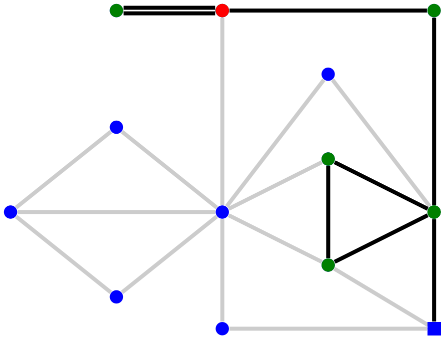

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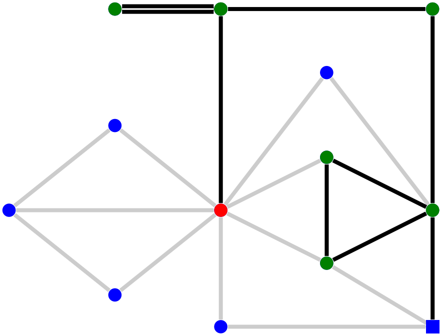

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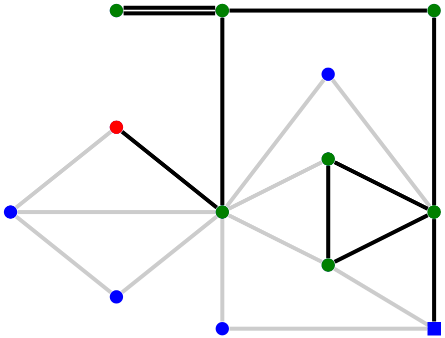

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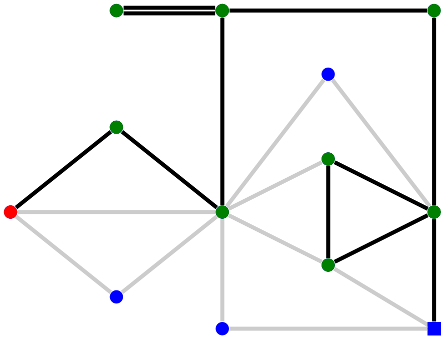

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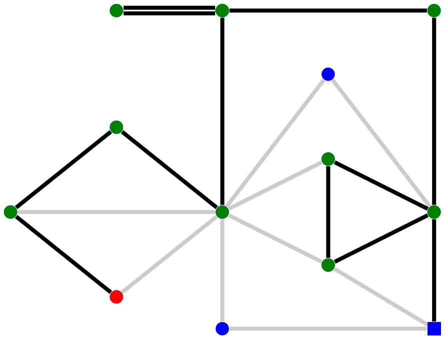

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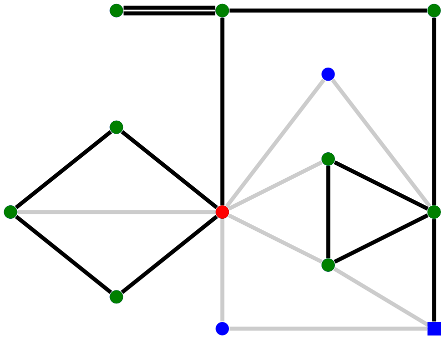

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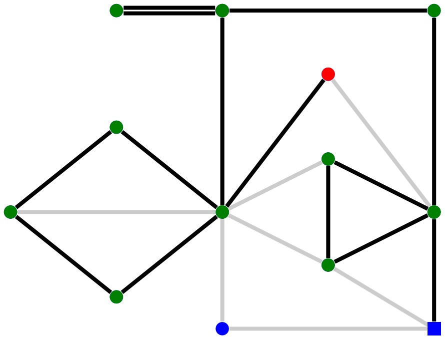

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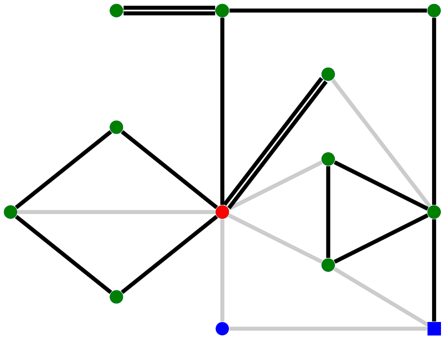

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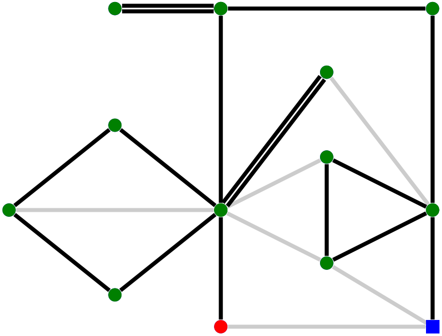

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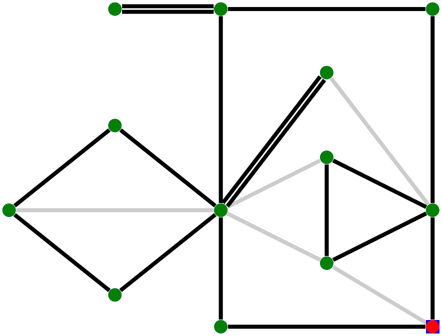

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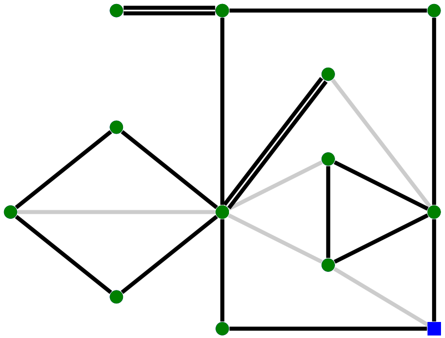

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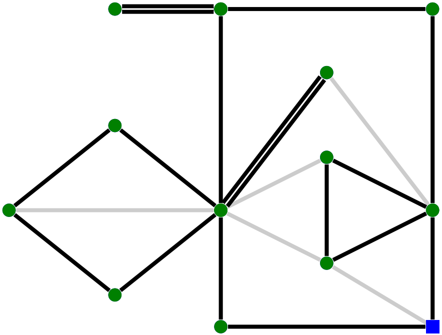

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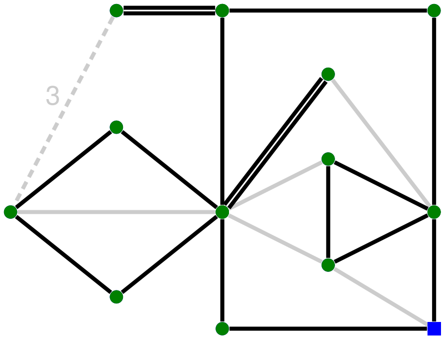

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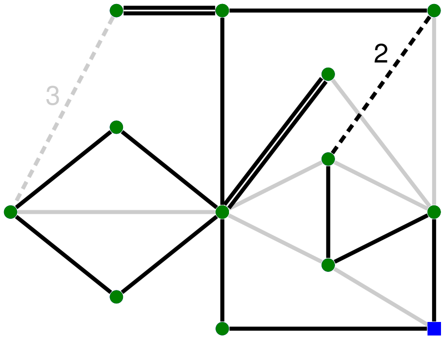

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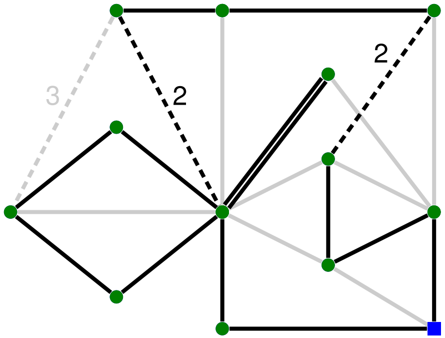

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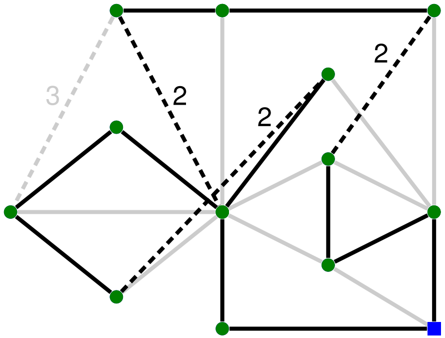

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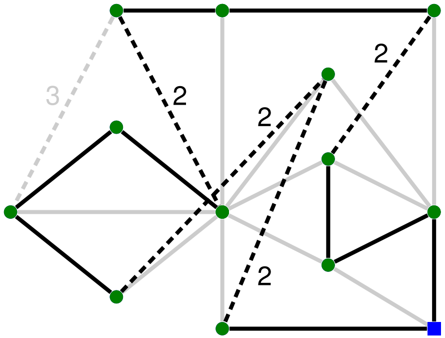

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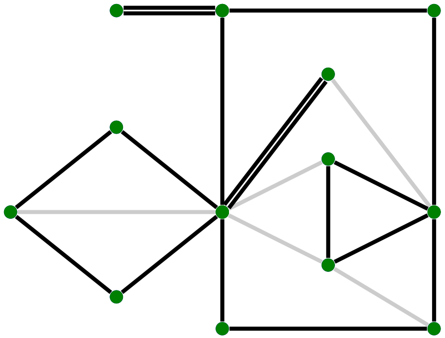

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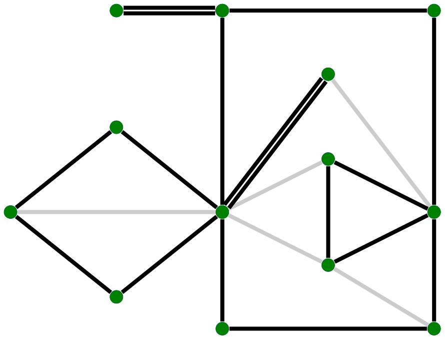

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Asymmetric TSP

Given a finite set V of cities and distances $c : V \times V \to \mathbb{R}_{\geq 0}$, find a tour (a list $v_0, \ldots, v_k$ containing each vertex at least once, with $v_0 = v_k$) of minimum total length $\sum_{i=1}^{k} c(v_{i-1}, v_i)$.

- $O(\log n)$-approximation algorithm, where $n = |V|$
  (Frieze, Galbiati, Maffioli [1982])
- $O(\log n / \log\log n)$-approximation algorithm
  (Asadpour, Goemans, Mądry, Oveis Gharan, Saberi [2010])
- no $\frac{75}{74}$-approximation algorithm exists unless $P = NP$
  (Karpinski, Lampis, Schmied [2013])
- integrality ratio between 2 and $\log^{O(1)} \log n$
  (Anari and Oveis Gharan [2015])

Essentially the same holds for the version where start $\neq$ end

Symmetric TSP: $c(v, w) = c(w, v)$ for all $v, w \in V$

- best known approximation ratio $\frac{3}{2}$
  (Christofides [1976])
- no $\frac{123}{122}$-approximation algorithm exists unless $P = NP$
  (Karpinski, Lampis, Schmied [2013])
- integrality ratio of subtour relaxation between $\frac{4}{3}$ and $\frac{3}{2}$
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Subtour relaxation (assuming triangle inequality):

$$\min \left\{ c(x) \;:\; x(\delta(v)) = 2\,(v \in V),\; x(\delta(U)) \geq 2\,(\emptyset \neq U \subset V),\; x \geq 0 \right\}$$

(Dantzig, Fulkerson, Johnson [1954], Held, Karp [1970],
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$$\sum_{e=\{v,w\}\in\binom{V}{2}} c(v,w)x_e \qquad \sum_{v\in U,\, w\in V\setminus U} x_{\{v,w\}}$$

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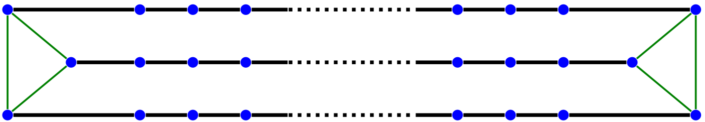


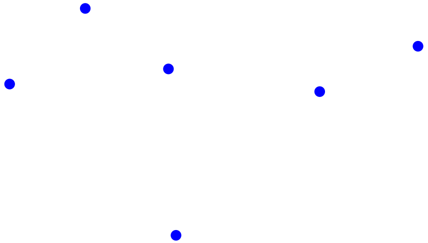
$c =$ graph distance

$x_e = \frac{1}{2}$

$x_e = 1$

Christofides' Algorithm

(Christofides [1976])



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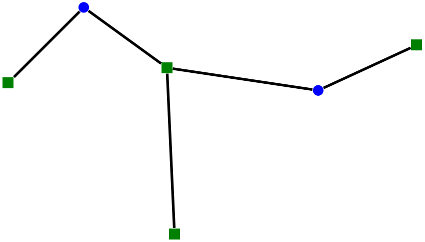

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- Take a cheapest spanning tree (V, S)
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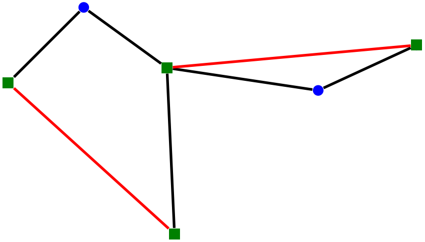

A T-join is a set J of edges such that T is the set of vertices with odd degree in (V, J). (Edmonds [1965])

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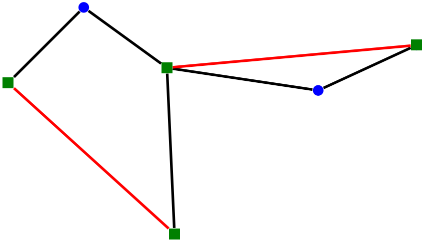

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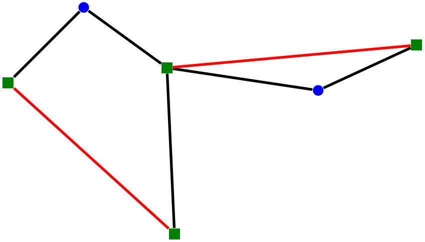

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Analysis:

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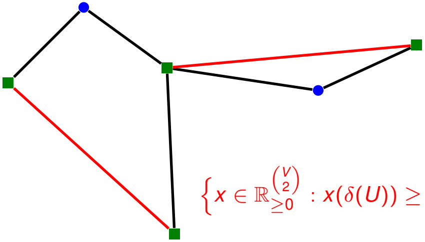

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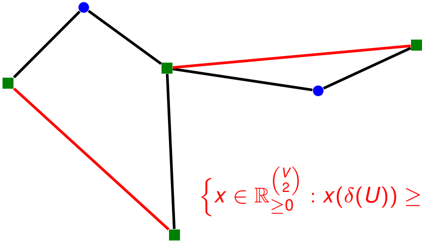

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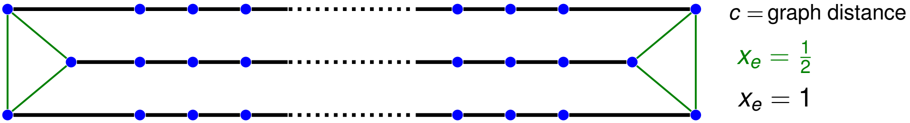
- $c(S + J) = c(S) + c(J) \leq \frac{n-1}{n} c(x^*) + \frac{1}{2} c(x^*) \leq \frac{3}{2} c(x^*)$.

Integrality ratio

Worst ratio of best integral solution (= optimum tour) and fractional solution (LP optimum)

- Wolsey's analysis shows an upper bound of $\frac{3}{2}$.
- These instances (of the Graph TSP) show a lower bound of $\frac{4}{3}$:



No better bounds are known!

Graph TSP

Given a graph $G = (V, E)$, let $c(v, w) =$ distance of v and w in G

Equivalently, look for a smallest Eulerian spanning subgraph of $2G$.

Improved approximation ratio for subcubic graphs:

- $\frac{4}{3}$ (Mömke, Svensson [2011])
  (before, for cubic graphs: (Boyd, Sitters, van der Ster, Stougie [2011]))
- $\frac{685}{684}$ impossible unless $P = NP$ (Karpinski, Schmied [2013])

Improved approximation ratios for general graphs:

- $1.5 - \epsilon$ (Oveis Gharan, Saberi, Singh [2011])
- 1.461 (Mömke, Svensson [2011])
- 1.445 (Mucha [2012])
- 1.4 (Sebő, V. [2014])

s-t-path TSP

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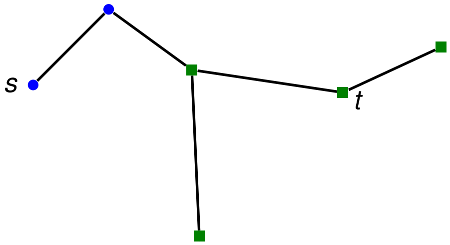

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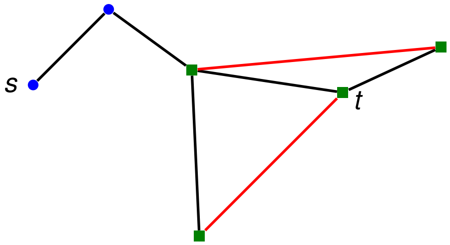
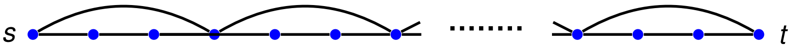


- take cheapest spanning tree S
- add cheapest T_S-join J

Lower bounds for the s-t-path TSP

Approximation ratio of Christofides/Hoogeveen is at least $\frac{5}{3}$:



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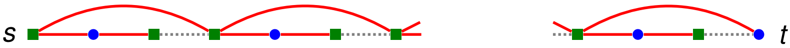

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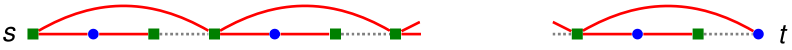
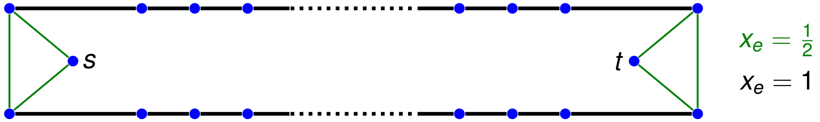


Integrality ratio is at least $\frac{3}{2}$:



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\end{array}
$$

s-t-path TSP: approximation ratios

General symmetric weights:

- 1.667 (Hoogeveen [1991])
- 1.619 (An, Kleinberg, Shmoys [2012])
- 1.6 (Sebő [2013])
- 1.599 (V. [2015])
- 1.566 (Gottschalk, V. [2016]) $\longleftarrow$
- 1.53 (Sebő, van Zuylen [2016]

In graphs:

- 1.586 (Mömke, Svensson [2011])
- 1.584 (Mucha [2012])
- 1.578 (An, Kleinberg, Shmoys [2012])
- 1.5 (Sebő, V. [2014]) $\longleftarrow$

Ear-decompositions

Write $G = P_0 + P_1 + \cdots + P_k$, where P_0 is a single vertex, and each P_i ($i = 1, \ldots, k$) is either

- a circuit sharing exactly one vertex with $P_0 + \cdots + P_{i-1}$, or
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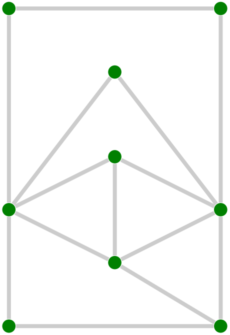

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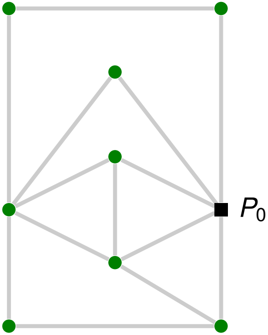

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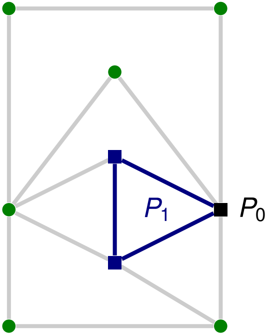

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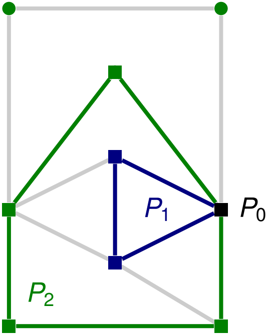

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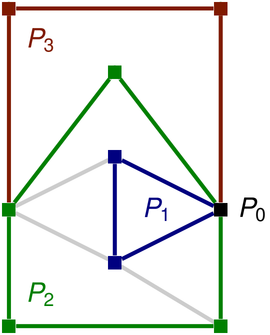

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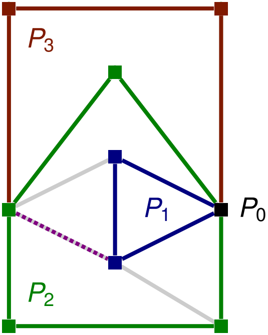

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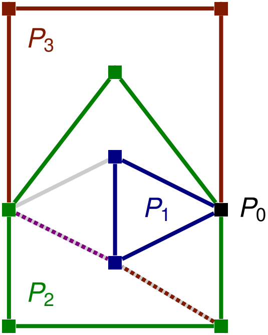

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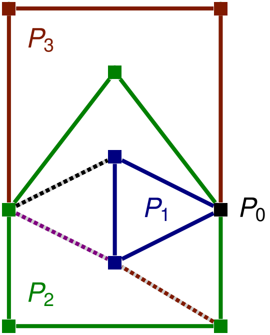

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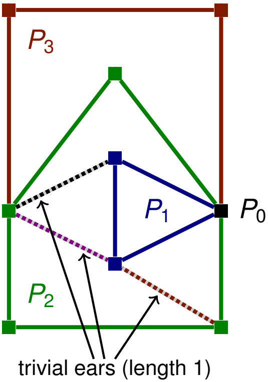

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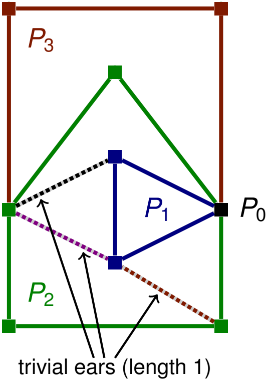

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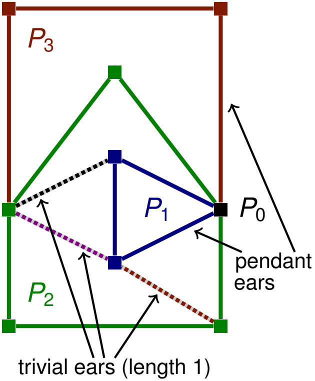

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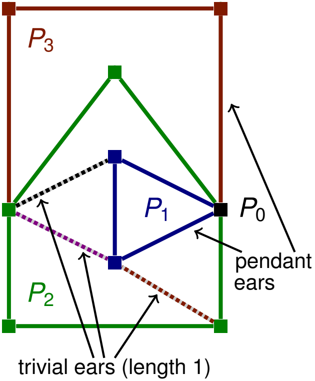

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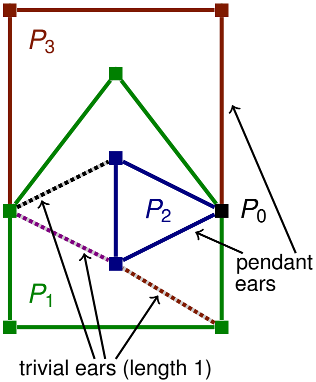

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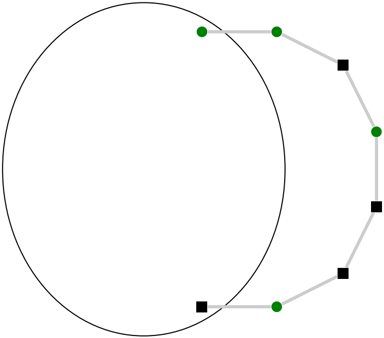
Ear induction for parity correction

For every T,

$$\min\{|J| : J \text{ is a } T\text{-join}\} \leq \frac{1}{2}(n - 1 + k_{\text{even}}),$$

where k_{even} is the number of even ears.



Proof:

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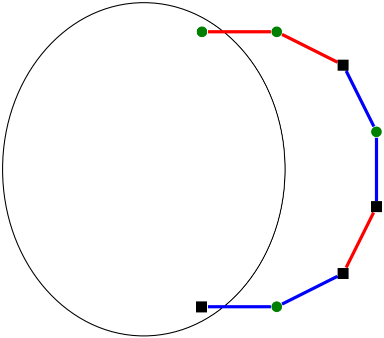

Proof:

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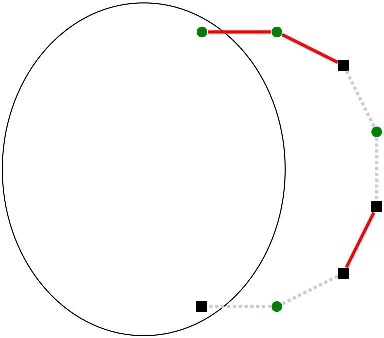

$\text{in}(P) :=$ number of inner vertices of P

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Proof:

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- Take the smaller part, with $\le \frac{1}{2}(\text{in}(P) + k_{\text{even}}(P))$ edges

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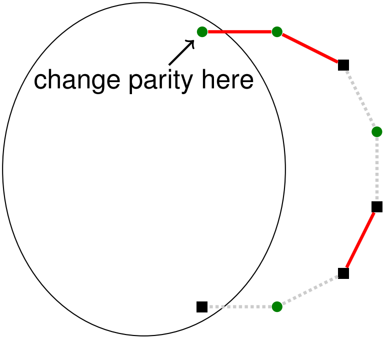

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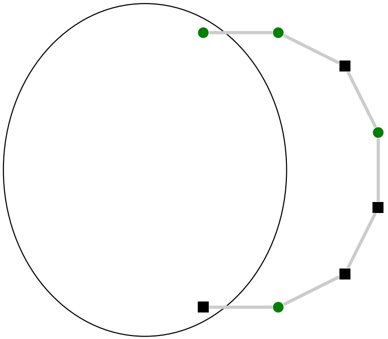
- Change parity of an endpoint of P if necessary; delete P; iterate $\square$

Ear induction for tours

Compute a tour with at most $\frac{3}{2}(n-1)+\frac{1}{2}\left(k_2-k_{\geq 4}\right)$ edges:



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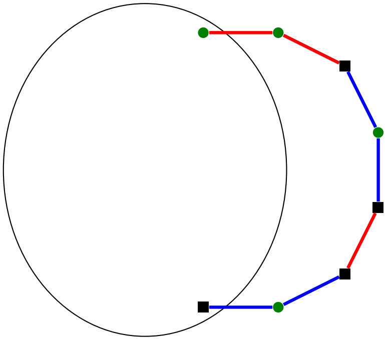

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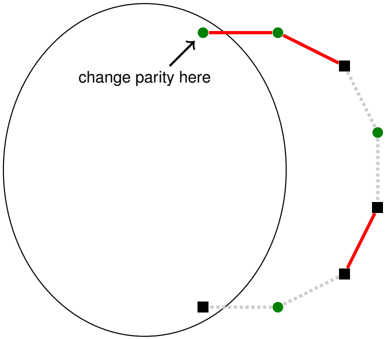

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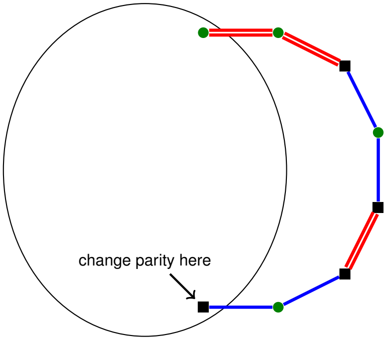

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- Take smaller part for obtaining a T-join.
- Double smaller part for obtaining a T-tour.

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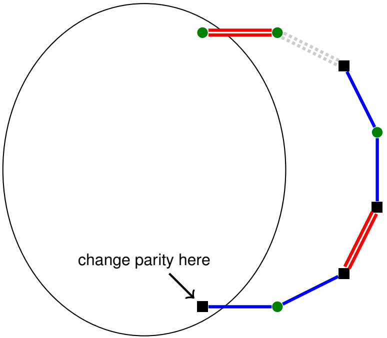

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- Take smaller part for obtaining a T-join.
- Double smaller part for obtaining a T-tour.
- May delete one pair of parallel edges (if there is one).

Need at most $\frac{3}{2}|\mathrm{in}(P)| - 1 + \frac{1}{2}k_{\mathrm{even}}(P)$ edges, or $\mathrm{in}(P) + 1$.
This is at most $\frac{3}{2}|\mathrm{in}(P)| + \frac{1}{2}\left(k_2(P) - k_{\geq 4}(P)\right)$. $\square$

Sketch of the first $\frac{3}{2}$-approximation algorithm for s-t-path TSP in graphs (Sebő, V. [2014])

- Compute an ear-decomposition in which the 2-ears are pendant and form a forest (using matroid intersection).
- If this is impossible, use Rado's theorem to get a stronger lower bound (details omitted).

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- (2) Take the 2-ears (but only one edge if s or t is the middle vertex), add edges for connectivity, and do parity correction. Yields a tour with at most $n - 1 + \frac{1}{2}\left(n - k_2 - 1 + k_{\geq 4}\right)$ edges. Good if $k_2 \geq k_{\geq 4}$.
- The better of the two tours has at most $\frac{3}{2}(n-1)$ edges.

LP relaxation for s-t-path TSP

$$
\begin{array}{llcll}
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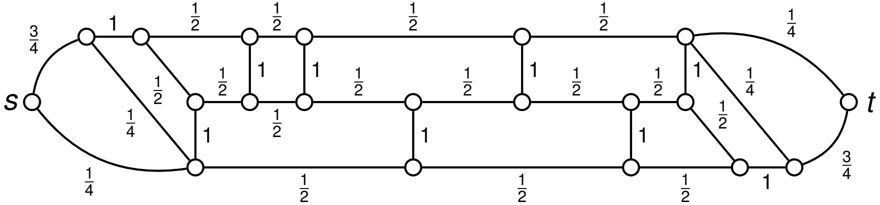

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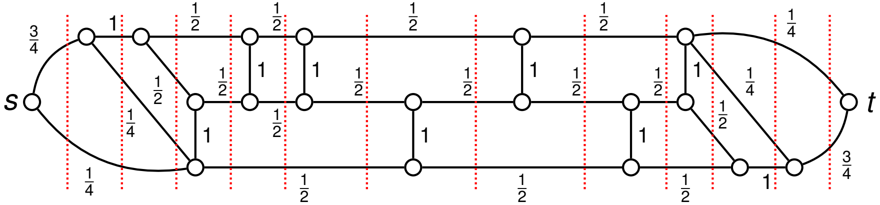
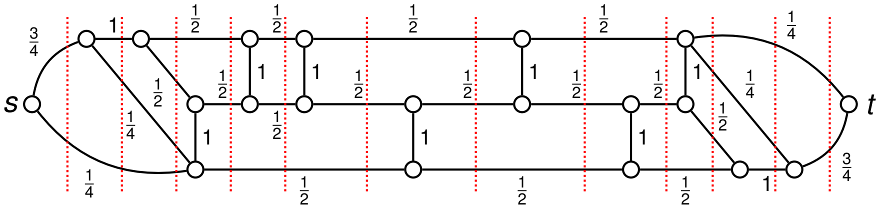


Cuts $C = \delta(U)$ with $x^*(C) < 2$ are called *narrow*. They form a chain.

Second $\frac{3}{2}$-approximation algorithm for s-t-path TSP in graphs

(Gao [2013])

- Solve the LP. Let x^* be an optimum solution.
- Gao's Theorem: There is a spanning tree (V, S) in the support that contains only one edge in every narrow cut. ("Gao tree")



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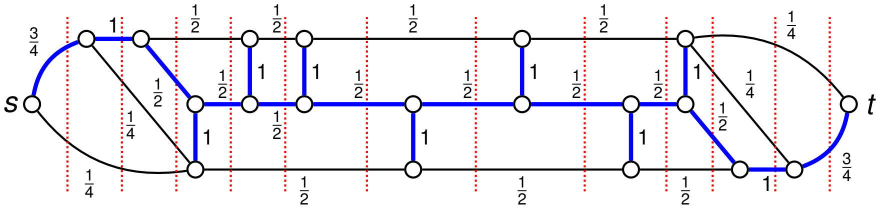

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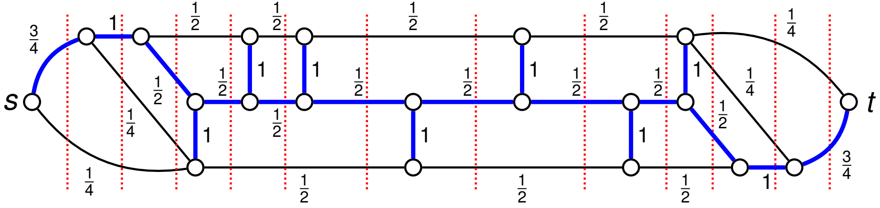
- Do parity correction. This costs at most $\frac{1}{2}c(x^*)$, because $\frac{1}{2}x^*$ dominates a vector in the convex hull of T_S-joins, where T_S is the set of vertices whose degree in S has the wrong parity
- Total number of edges at most $n - 1 + \frac{1}{2}c(x^*)$.



Now: general metrics

Most new algorithms (for all TSP variants) for general metrics

- first solve the natural LP relaxation,
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Different distributions were used:

- max entropy distribution (Asadpour et al. [2010], Oveis Gharan et al. [2011])
- arbitrary distribution (An, Kleinberg, Shmoys [2012], Sebő [2013])
- distribution improved by local reassembling (V. [2015])
- Gao tree distribution (Gottschalk, V. [2016])

Best-of-Many-Christofides (An, Kleinberg, Shmoys [2012])

- Solve the LP, let x^* be an optimum solution
- Decompose x^* into spanning trees: write

$$x^* = \sum_{S \in \mathcal{S}} p_S \chi^S$$

  where $p_S \geq 0$ ($S \in \mathcal{S}$) and $\sum_{S \in \mathcal{S}} p_S = 1$
- Do parity correction for each $S \in \mathcal{S}$ with $p_S > 0$: add a minimum cost T_S-join
- Take the best of these tours

$\mathcal{S}$ is the set of edge sets of spanning trees

T_S is the set of vertices whose degree in S has the wrong parity (even for s or t, odd for other vertices)

Basic analysis

(An, Kleinberg, Shmoys [2012])

The result has cost

$$
\begin{aligned}
& \min_{S \in \mathcal{S}:\, p_S > 0} \big(c(S) + \min\{c(J) : J \text{ is a } T_S\text{-join}\} \big) \\
\leq \;& \sum_{S \in \mathcal{S}} p_S \big(c(S) + \min\{c(J) : J \text{ is a } T_S\text{-join}\} \big) \\
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\Big\{ y \in \mathbb{R}^E_{\geq 0} : y(C) \geq 1 \ \forall\ T_S\text{-cuts } C \Big\}
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Example: x^* is a correction vector for every S, and $\frac{x^*}{2}$ almost

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Fact: narrow cut C is a T_S-cut $\Leftrightarrow$ $|S \cap C|$ even

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- Need a correction vector y^S with $y^S(C) \geq 1$ for all T_S-cuts C.
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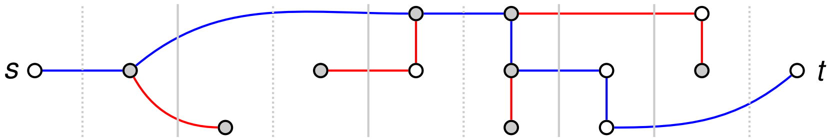

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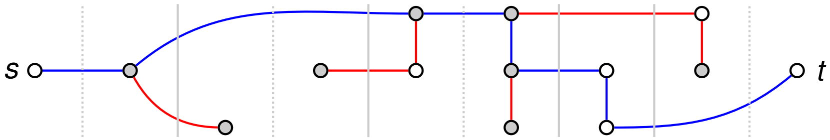

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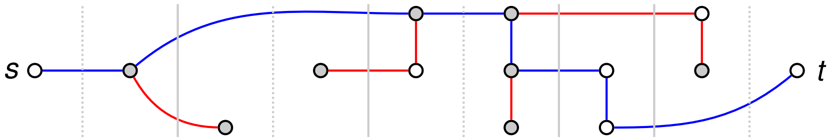

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- Gao trees do not need fixing.
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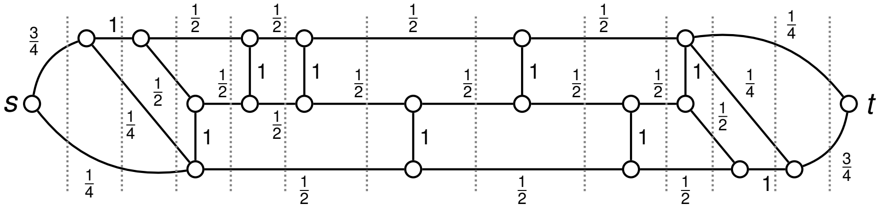

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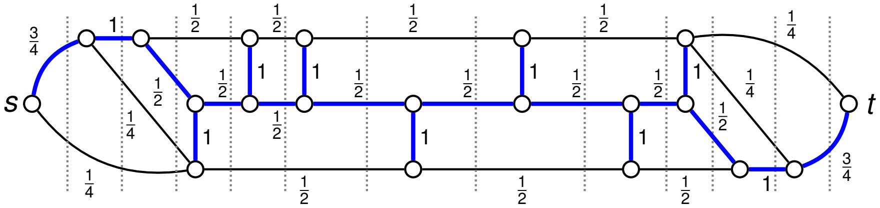

S_1 with $p_1 = \frac{1}{2}$

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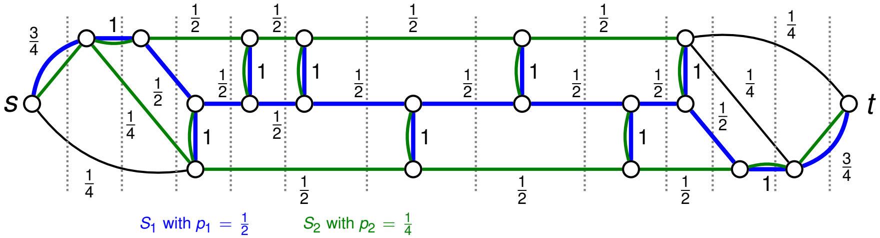

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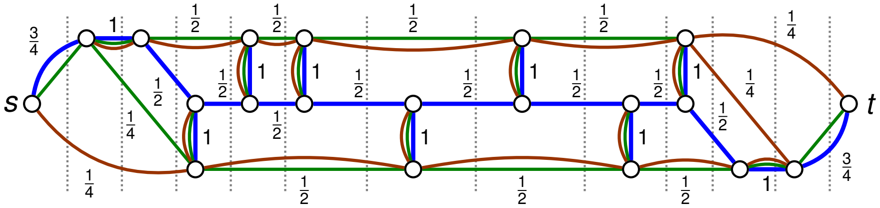

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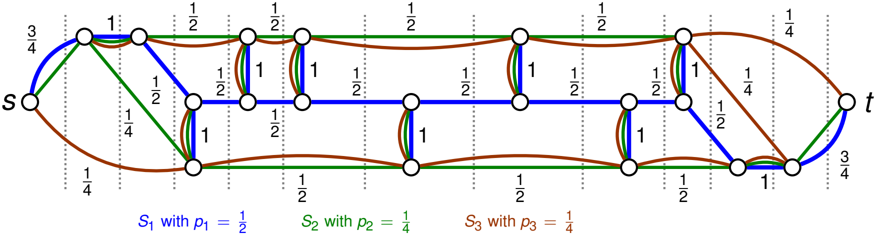

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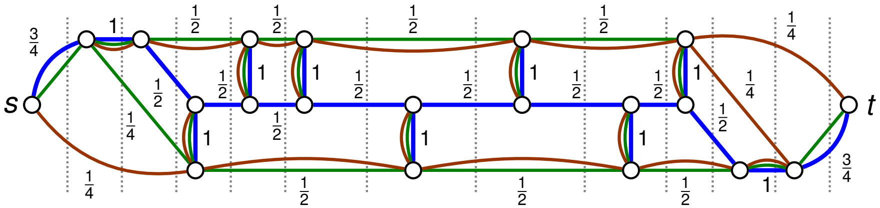

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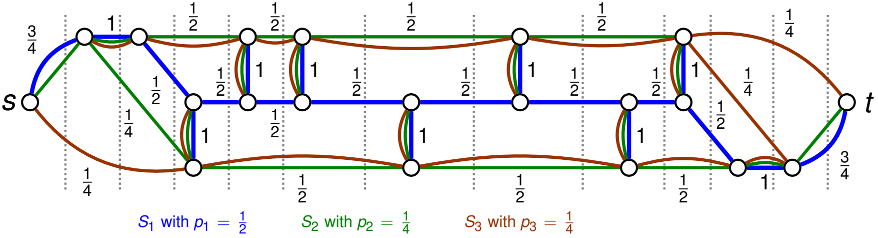

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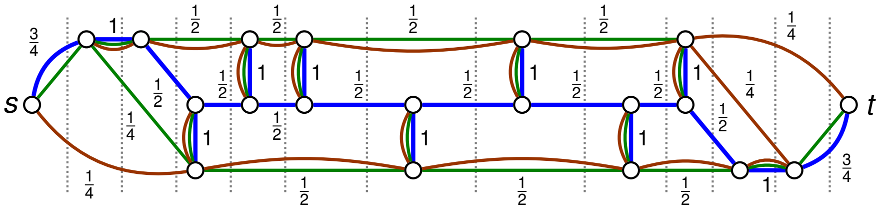

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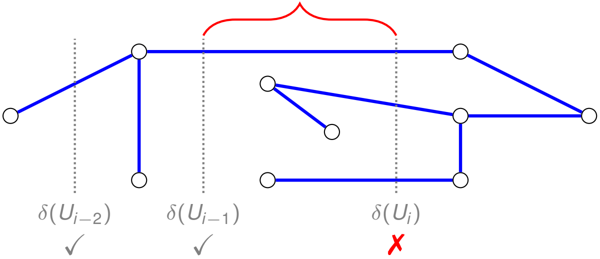
- can be computed in polynomial time
- yields approximation ratio 1.566 (with best-of-many)
- also used by Sebő and van Zuylen [2016] for ratio $\frac{26}{17}$

Proof: outline

- Start with
$$x^* = \frac{1}{r}\sum_{j=1}^{r}\chi^{S_j}$$
- Deal with the trees $S_j \ (j = 1, \ldots, r)$ in this order
- For each j: let
$$\{s\} = U_1 \subset \cdots \subset U_k = V \setminus \{t\}$$
be the sets with
$$x^*(\delta(U_i)) \leq 2 - \frac{j}{r}$$
- Deal with the cuts $\delta(U_i) \ (i = 1, \ldots, k)$ in this order
- Need $|S_j \cap \delta(U_i)| = 1$
- First make $S_j[U_i]$ connected

Proof: make $S_j[U_i]$ connected

Case 1: $S_j[U_i \setminus U_{i-1}]$ disconnected



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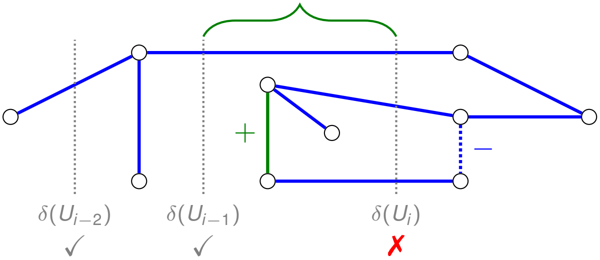

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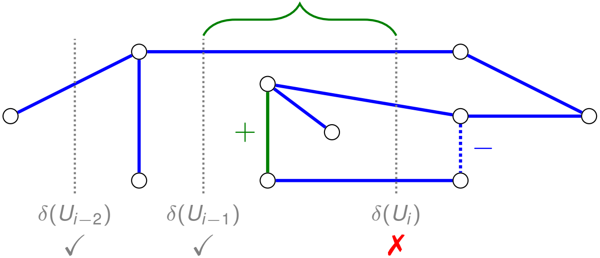

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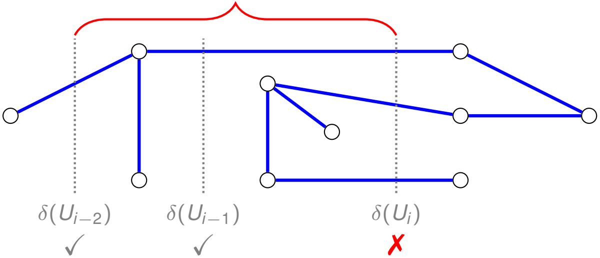

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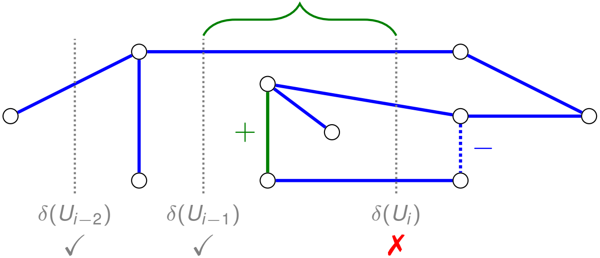
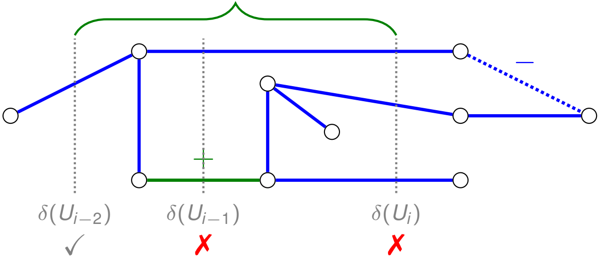


Case 2: $S_j[U_i \setminus U_{i-1}]$ connected $\Rightarrow \exists\, k > j : S_k \cap \delta(U_{i-1}) \cap \delta(U_i) = \emptyset$



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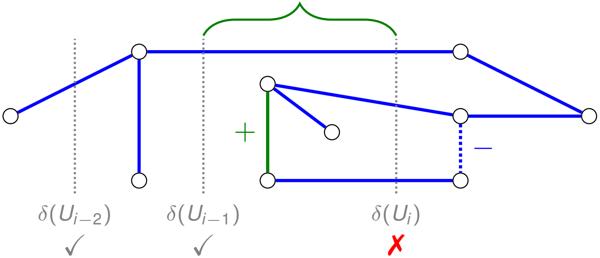

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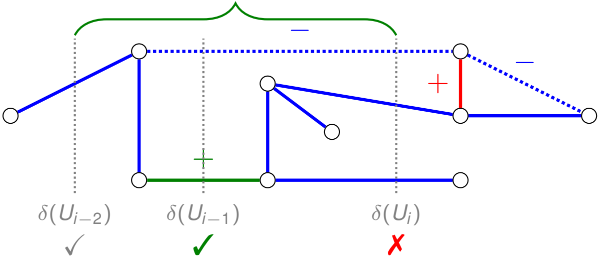

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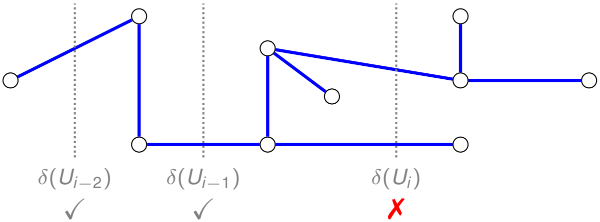

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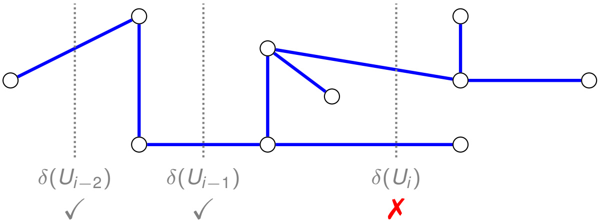

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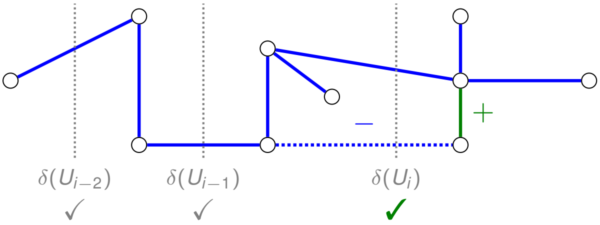

TSP variants – state of the art

Integrality ratios. Upper bounds = approximation ratios unless mentioned otherwise

2ECSS, general metrics:

- between $\frac{6}{5}$ and $\frac{3}{2}$ (Alexander, Boyd, Elliott-Magwood [2006])

2ECSS, unweighted graphs:

- between $\frac{8}{7}$ (Boyd, Fu, Sun [2014]) and $\frac{4}{3}$ (Sebő, V. [2014])

TSP, general metrics:

- between $\frac{4}{3}$ and $\frac{3}{2}$ (Wolsey [1980])

TSP, unweighted graphs:

- between $\frac{4}{3}$ and $\frac{7}{5}$ (Sebő, V. [2014])

s-t-path TSP, general metrics:

- between $\frac{3}{2}$ and $\frac{26}{17}$ (Sebő, van Zuylen [2016])

s-t-path TSP, unweighted graphs:

- $\frac{3}{2}$ (Sebő, V. [2014])

ATSP, $\triangle$-inequality:

- between 2 (Boyd, Elliott-Magwood [2005], Charikar, Goemans, Karloff [2006])
  and $\log^{O(1)} \log n$ (Anari and Oveis Gharan [2015]);
  apx ratio $8 \log n / \log \log n$ (Asadpour, Goemans, Mądry, Oveis Gharan, Saberi [2010])

ATSP, unweighted digraphs:

- between $\frac{3}{2}$ (Gottschalk [2013]) and 13; apx ratio $27 + \epsilon$ (Svensson [2015])

How important are integrality ratios?

- We cannot solve the LPs combinatorially in polynomial time.
- Integrality ratios do not imply lower bounds on approximability.

Example: Euclidean TSP

- approximation scheme (Arora [1998])
- subtour relaxation has integrality ratio $\frac{4}{3}$ (Hougardy [2014])

- Integrality ratios imply bounds on what we can achieve if we use this LP as lower bound.

Current and future research

- better than $\frac{3}{2}$ for s-t-path TSP in graphs?
- $\frac{3}{2}$ for s-t-path TSP in general metrics?
- better than Sebő's $\frac{8}{5}$ for T-tours in general metrics?
- improve on Christofides' $\frac{3}{2}$-approximation algorithm? $\frac{4}{3}$?
- constant factor for asymmetric TSP?
- generalizations and practical applications

Thank you!

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