Preface to the Fifth Edition

When preparing the first edition of this book, more than ten years ago, we tried to accomplish two objectives: it should be useful as an advanced graduate textbook, but also as a reference work for research. With each new edition we have to decide how the book can be improved further. Of course, it is less and less possible to describe the growing area comprehensively.

If we included everything that we like, the book would grow beyond a single volume. Since the book is used for many courses, now even sometimes at undergraduate level, we thought that adding some classical material might be more useful than including a selection of the latest results.

In this edition, we added a proof of Cayley's formula, more details on blocking flows, the new faster b-matching separation algorithm, an approximation scheme for multidimensional knapsack, and results concerning the multicommodity max-flow min-cut ratio and the sparsest cut problem. There are further small improvements in numerous places and more than 60 new exercises. Of course, we also updated the references to point to the most recent results and corrected some minor errors that were discovered.

We would like to thank Takao Asano, Maxim Babenko, Ulrich Brenner, Benjamin Bolten, Christoph Buchheim, Jean Fonlupt, András Frank, Michael Gester, Stephan Held, Stefan Hougardy, Hiroshi Iida, Klaus Jansen, Alexander Karzanov, Levin Keller, Alexander Kleff, Niko Klewinghaus, Stefan Knauf, Barbara Langfeld, Jens Maßberg, Marc Pfetsch, Klaus Radke, Rabe von Randow, Tomás Salles, Jan Schneider, Christian Schulte, András Sebő, Martin Skutella, Jácint Szabó, and Simon Wedeking for valuable feedback on the previous edition.

We are pleased that this book has been received so well, and further translations are on their way. Editions in Japanese, French, Italian, German, Russian, and Chinese have appeared since 2009 or are scheduled to appear soon. We hope that our book will continue to serve its purpose in teaching and research in combinatorial optimization.

Bonn, September 2011 *Bernhard Korte and Jens Vygen*

Preface to the Fourth Edition

With four English editions, and translations into four other languages forthcoming, we are very happy with the development of our book. Again, we have revised, updated, and significantly extended it for this fourth edition. We have added some classical material that may have been missed so far, in particular on linear programming, the network simplex algorithm, and the max-cut problem. We have also added a number of new exercises and up-to-date references. We hope that these changes serve to make our book an even better basis for teaching and research.

We gratefully acknowledge the continuous support of the Union of the German Academies of Sciences and Humanities and the NRW Academy of Sciences via the long-term research project "Discrete Mathematics and Its Applications". We also thank those who gave us feedback on the third edition, in particular Takao Asano, Christoph Bartoschek, Bert Besser, Ulrich Brenner, Jean Fonlupt, Satoru Fujishige, Marek Karpinski, Jens Maßberg, Denis Naddef, Sven Peyer, Klaus Radke, Rabe von Randow, Dieter Rautenbach, Martin Skutella, Markus Struzyna, Jürgen Werber, Minyi Yue, and Guochuan Zhang, for their valuable comments. At `http://www.or.uni-bonn.de/~vygen/co.html` we will continue to maintain updated information about this book.

Bonn, August 2007 *Bernhard Korte and Jens Vygen*

Preface to the Third Edition

After five years it was time for a thoroughly revised and substantially extended edition. The most significant feature is a completely new chapter on facility location. No constant-factor approximation algorithms were known for this important class of *NP*-hard problems until eight years ago. Today there are several interesting and very different techniques that lead to good approximation guarantees, which makes this area particularly appealing, also for teaching. In fact, the chapter has arisen from a special course on facility location.

Many of the other chapters have also been extended significantly. The new material includes Fibonacci heaps, Fujishige's new maximum flow algorithm, flows over time, Schrijver's algorithm for submodular function minimization, and the Robins-Zelikovsky Steiner tree approximation algorithm. Several proofs have been streamlined, and many new exercises and references have been added.

We thank those who gave us feedback on the second edition, in particular Takao Asano, Yasuhito Asano, Ulrich Brenner, Stephan Held, Tomio Hirata, Dirk Müller, Kazuo Murota, Dieter Rautenbach, Martin Skutella, Markus Struzyna and Jürgen Werber, for their valuable comments. Eminently, Takao Asano's notes and Jürgen Werber's proofreading of Chapter 22 helped to improve the presentation at various places.

Again we would like to mention the Union of the German Academies of Sciences and Humanities and the Northrhine-Westphalian Academy of Sciences. Their continuous support via the long-term project "Discrete Mathematics and Its Applications" funded by the German Ministry of Education and Research and the State of Northrhine-Westphalia is gratefully acknowledged.

Bonn, May 2005 *Bernhard Korte and Jens Vygen*

Preface to the Second Edition

It was more than a surprise to us that the first edition of this book already went out of print about a year after its first appearance. We were flattered by the many positive and even enthusiastic comments and letters from colleagues and the general readership. Several of our colleagues helped us in finding typographical and other errors. In particular, we thank Ulrich Brenner, András Frank, Bernd Gärtner and Rolf Möhring. Of course, all errors detected so far have been corrected in this second edition, and references have been updated.

Moreover, the first preface had a flaw. We listed all individuals who helped us in preparing this book. But we forgot to mention the institutional support, for which we make amends here.

It is evident that a book project which took seven years benefited from many different grants. We would like to mention explicitly the bilateral Hungarian-German Research Project, sponsored by the Hungarian Academy of Sciences and the Deutsche Forschungsgemeinschaft, two Sonderforschungsbereiche (special research units) of the Deutsche Forschungsgemeinschaft, the Ministère Français de la Recherche et de la Technologie and the Alexander von Humboldt Foundation for support via the Prix Alexandre de Humboldt, and the Commission of the European Communities for participation in two projects DONET. Our most sincere thanks go to the Union of the German Academies of Sciences and Humanities and to the Northrhine-Westphalian Academy of Sciences. Their long-term project "Discrete Mathematics and Its Applications" supported by the German Ministry of Education and Research (BMBF) and the State of Northrhine-Westphalia was of decisive importance for this book.

Bonn, October 2001 *Bernhard Korte and Jens Vygen*

Preface to the First Edition

Combinatorial optimization is one of the youngest and most active areas of discrete mathematics, and is probably its driving force today. It became a subject in its own right about 50 years ago.

This book describes the most important ideas, theoretical results, and algorithms in combinatorial optimization. We have conceived it as an advanced graduate text which can also be used as an up-to-date reference work for current research. The book includes the essential fundamentals of graph theory, linear and integer programming, and complexity theory. It covers classical topics in combinatorial optimization as well as very recent ones. The emphasis is on theoretical results and algorithms with provably good performance. Applications and heuristics are mentioned only occasionally.

Combinatorial optimization has its roots in combinatorics, operations research, and theoretical computer science. A main motivation is that thousands of real-life problems can be formulated as abstract combinatorial optimization problems. We focus on the detailed study of classical problems which occur in many different contexts, together with the underlying theory.

Most combinatorial optimization problems can be formulated naturally in terms of graphs and as (integer) linear programs. Therefore this book starts, after an introduction, by reviewing basic graph theory and proving those results in linear and integer programming which are most relevant for combinatorial optimization.

Next, the classical topics in combinatorial optimization are studied: minimum spanning trees, shortest paths, network flows, matchings and matroids. Most of the problems discussed in Chapters 6–14 have polynomial-time ("efficient") algorithms, while most of the problems studied in Chapters 15–21 are *NP*-hard, i.e. a polynomial-time algorithm is unlikely to exist. In many cases one can at least find approximation algorithms that have a certain performance guarantee. We also mention some other strategies for coping with such "hard" problems.

This book goes beyond the scope of a normal textbook on combinatorial optimization in various aspects. For example we cover the equivalence of optimization and separation (for full-dimensional polytopes), $O(n^3)$-implementations of matching algorithms based on ear-decompositions, Turing machines, the Perfect Graph Theorem, *MAXSNP*-hardness, the Karmarkar-Karp algorithm for bin packing, recent approximation algorithms for multicommodity flows, survivable network design and the Euclidean traveling salesman problem. All results are accompanied by detailed proofs.

Of course, no book on combinatorial optimization can be absolutely comprehensive. Examples of topics which we mention only briefly or do not cover at all are tree-decompositions, separators, submodular flows, path-matchings, delta-matroids, the matroid parity problem, location and scheduling problems, nonlinear problems, semidefinite programming, average-case analysis of algorithms, advanced data structures, parallel and randomized algorithms, and the theory of probabilistically checkable proofs (we cite the *PCP* Theorem without proof).

At the end of each chapter there are a number of exercises containing additional results and applications of the material in that chapter. Some exercises which might be more difficult are marked with an asterisk. Each chapter ends with a list of references, including texts recommended for further reading.

This book arose from several courses on combinatorial optimization and from special classes on topics like polyhedral combinatorics or approximation algorithms. Thus, material for basic and advanced courses can be selected from this book.

We have benefited from discussions and suggestions of many colleagues and friends and – of course – from other texts on this subject. Especially we owe sincere thanks to András Frank, László Lovász, András Recski, Alexander Schrijver and Zoltán Szigeti. Our colleagues and students in Bonn, Christoph Albrecht, Ursula Bünnagel, Thomas Emden-Weinert, Mathias Hauptmann, Sven Peyer, Rabe von Randow, André Rohe, Martin Thimm and Jürgen Werber, have carefully read several versions of the manuscript and helped to improve it. Last, but not least we thank Springer Verlag for the most efficient cooperation.

Bonn, January 2000 *Bernhard Korte and Jens Vygen*

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