Resource Sharing, Routing, Chip Design

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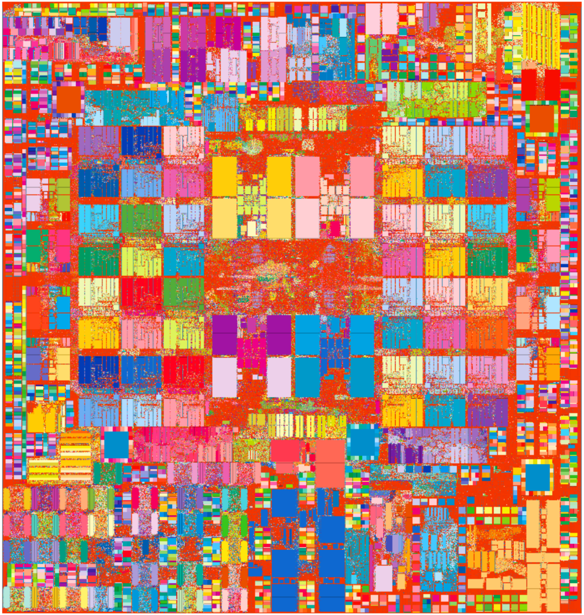
joint work with Markus Ahrens, Stephan Held, Niko Klewinghaus, Dirk Müller, Klaus Radke, Daniel Rotter, Pietro Saccardi, Rudolf Scheifele, Christian Schulte, Vera Traub, et al.

Chip design: placement

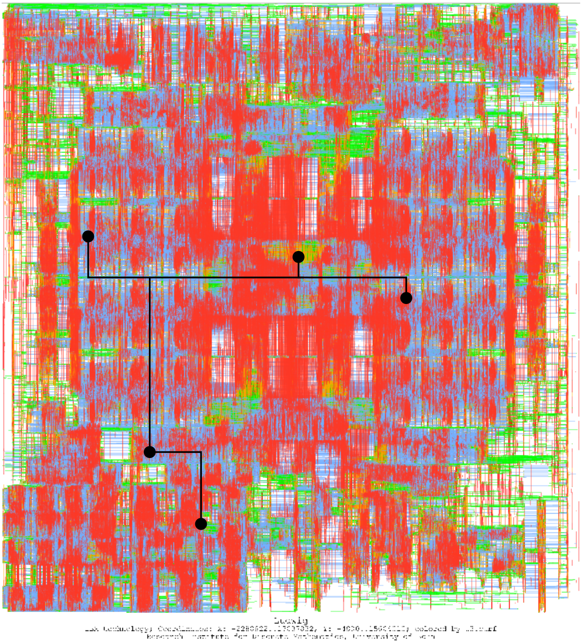


telecommunication chip with ≈

1 billion transistors

8 million circuits

Chip design: routing



9 routing layers

30 million pins

8 million nets

100 million
wire segments

100 million vias

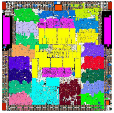
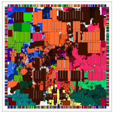
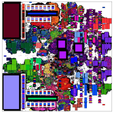
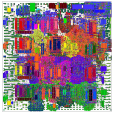
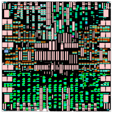
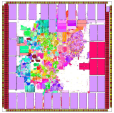
1 kilometer
total wire length

Combinatorial optimization in chip design

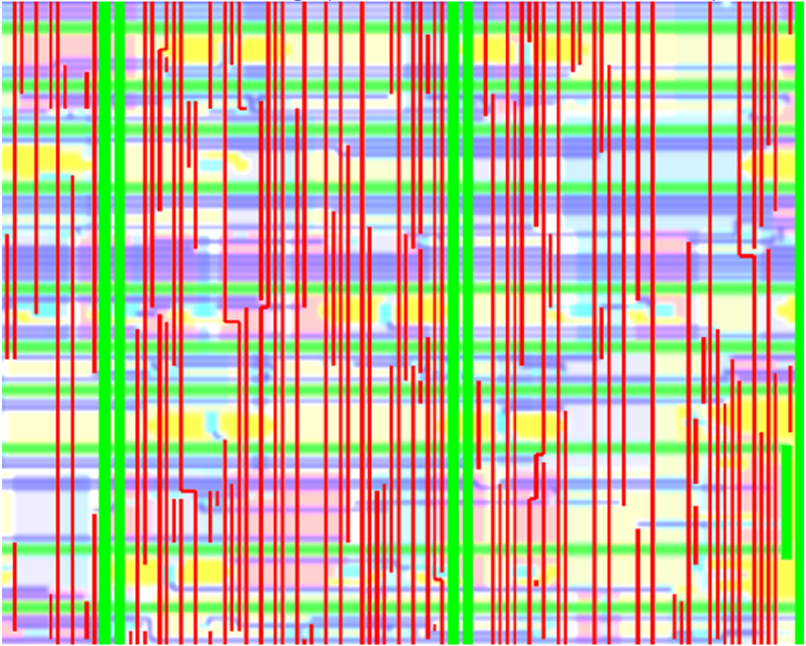
- Boolean functions and circuits
- shortest paths, TSP, spanning trees, Steiner trees
- rectangle packing, knapsack problem, bin packing
- facility location, partitioning, clustering
- maximum flows, discrete time-cost tradeoff problem
- minimum mean cycles, parametric shortest paths
- transportation and minimum cost flows
- multicommodity flows, disjoint paths and trees, resource sharing

The BonnTools,

- developed by our group at the University of Bonn,
- cover all major areas of layout and timing optimization,
- include libraries for combinatorial optimization, advanced data structures, computational geometry, etc.,
- have more than a million lines of C++ code,
- are being used worldwide by IBM and other companies,
- have been used for the design of thousands of chips,
- including several complete microprocessor series
- and the most complex chips of major technology companies.

Detail of the routing (less than one millionth)



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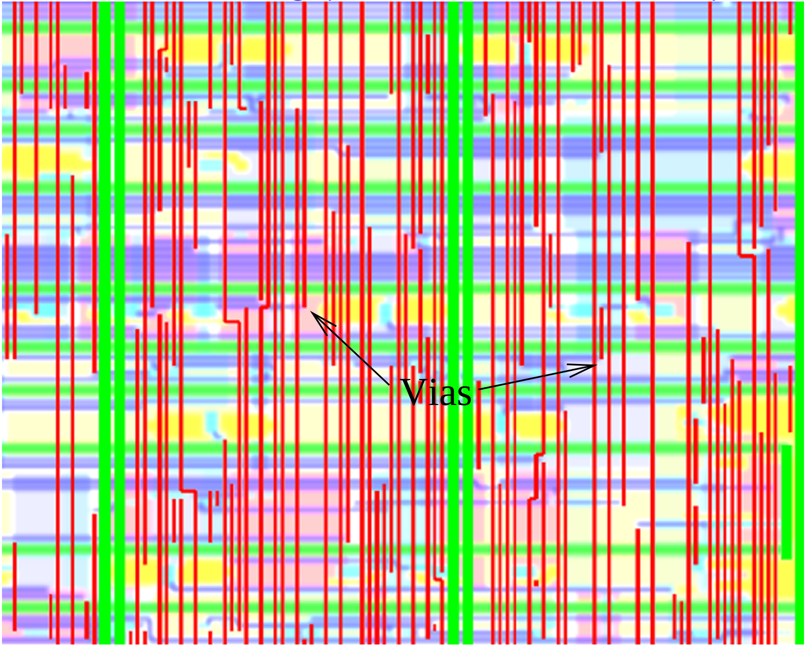

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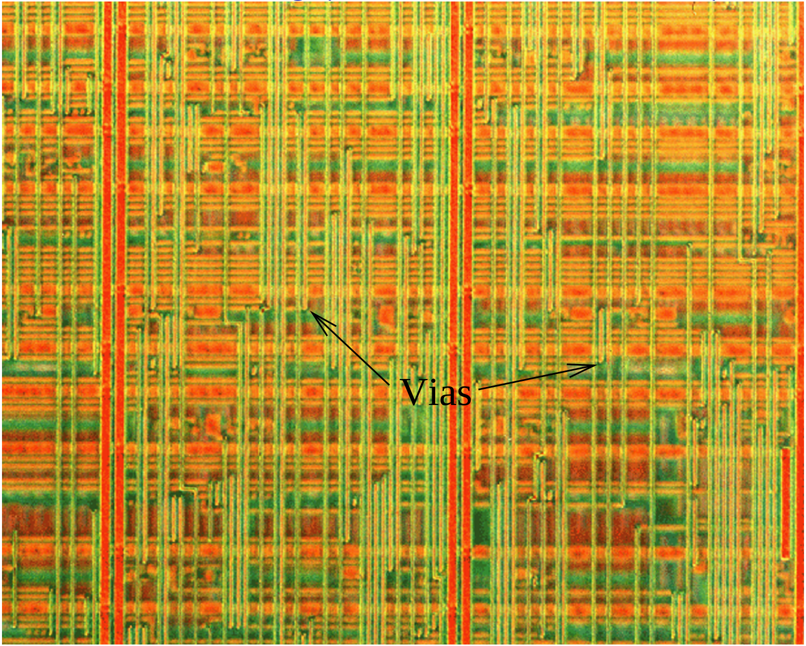

Routing: task

Instance:

- a number of routing planes
- a set of nets, where each net is a set of pins (terminals)
- a set of shapes for each pin
- a set of blockage shapes
- rules that tell if two given shapes are connected, separated

Task:

Compute a feasible routing, i.e.,
a set of wire shapes for each net, connecting the pins,
separate from blockages and shapes of other nets

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- a set of blockage shapes
- rules that tell if two given shapes are connected, separated
- timing constraints
- information on power consumption, yield, ...

Task:

Compute a feasible routing, i.e.,
a set of wire shapes for each net, connecting the pins,
separate from blockages and shapes of other nets

- such that all timing constraints are met
- and the (estimated) power consumption and/or manufacturing cost is minimized.

Routing: simplified view

Find vertex-disjoint Steiner trees connecting given terminal sets in a 3-dimensional grid graph.

NP-hard:
no polynomial-time algorithm unless $P = NP$

Order of magnitude:
10 million Steiner trees in a graph with 500 billion vertices
$\rightarrow$ Even linear-time algorithms are too slow!

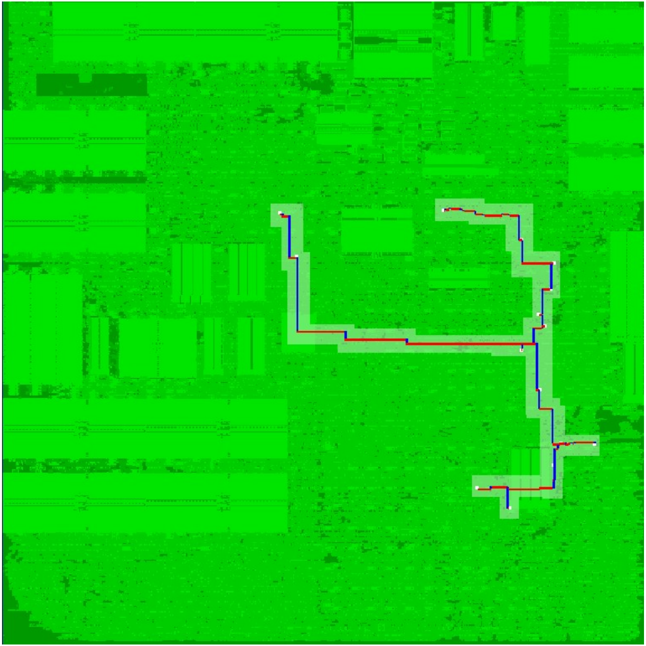
Global and detailed routing

Routing is usually performed in three phases:

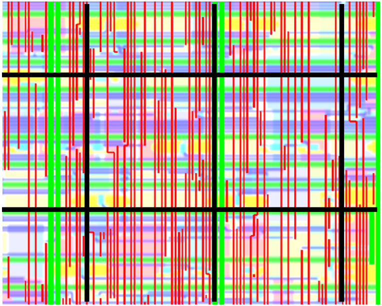
- Global routing: performs global optimization, determines a routing area (corridor) for each net, but no detailed view
- Detailed routing: constructs wires connecting each net within this corridor, respecting all design rules necessary for the lithographic process in fabrication
- Postoptimization: fix remaining violated constraints, improve the wiring by spreading and do some postprocessing for more robust manufacturing

Graphs are huge: 500 000 000 000 vertices in detailed routing, still 100 000 000 vertices in global routing.

Detailed routing in global routing corridor



Global routing: classical model



In each routing plane: contract regions of approx. 70x70 tracks to a single vertex

- compute capacities of edges between adjacent regions
- pack Steiner trees with respect to these edge capacities
- global optimization of objective functions
- Steiner tree yields routing corridor for each net
- Detailed routing computes detailed wires in these corridors by fast goal-oriented variants of Dijkstra's algorithm and optimal pin access (Hetzel [1998], Müller [2009], Peyer, Rautenbach, Vygen [2009], Klewinghaus [2013], Gester et al. [2013], Ahrens et al. [2015], Henke [2016], Ahrens, Rabenstein [2019])

Global routing: classical problem formulation

Instance:

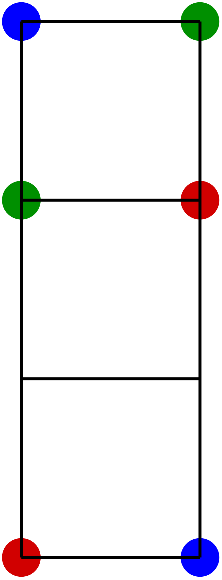
- a global routing (grid) graph with edge capacities
- a set of nets, each consisting of a set of vertices (terminals)

Task: find a Steiner tree for each net such that

- the edge capacities are respected,
- and (weighted) netlength is minimum.

Simple example

- edge-disjoint paths problem
- 3 terminal pairs: blue, red, green
- each terminal pair has demand 1
- each edge has capacity 1



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- edge-disjoint paths problem
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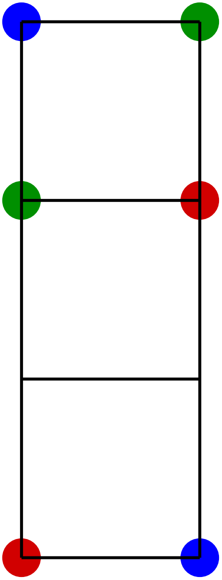
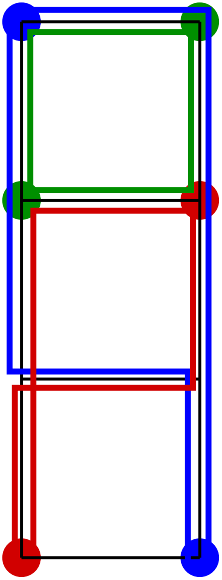
- no solution exists
- edge-disjoint paths problem is *NP*-hard (even in planar grids)



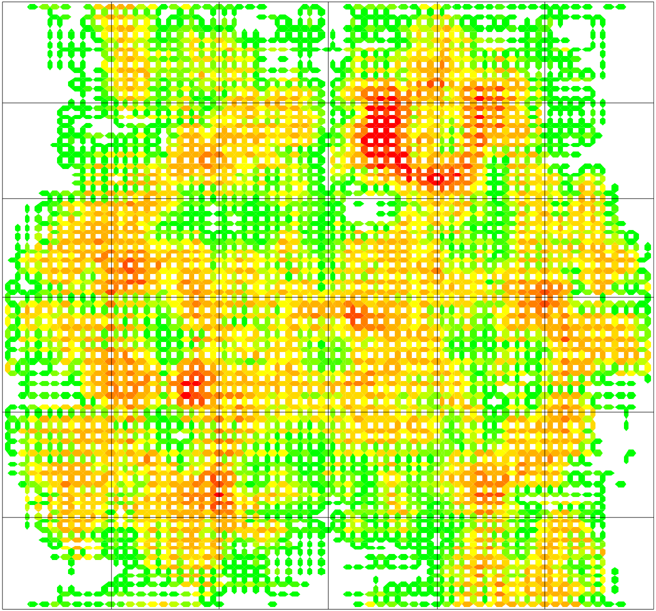
Simple example

- edge-disjoint paths problem
- 3 terminal pairs: blue, red, green
- each terminal pair has demand 1
- each edge has capacity 1
- no solution exists
- fractional solution exists
  (route $\frac{1}{2}$ along each colored path)



- edge-disjoint paths problem is *NP*-hard (even in planar grids)
- fractional relaxation can be solved in polynomial time by linear programming (but not fast enough)

Real example: global routing congestion map



Min-max resource sharing

Instance

- finite sets $\mathcal{R}$ of **resources** and $\mathcal{C}$ of **customers**
- for each $c \in \mathcal{C}$: a set $\mathcal{B}_c \subseteq \mathbb{R}^{\mathcal{R}}_{\geq 0}$ of **feasible solutions**

Task

- Find a $b_c \in \mathcal{B}_c$ for each $c \in \mathcal{C}$
  with minimum **congestion**

$$\max_{r \in \mathcal{R}} \sum_{c \in \mathcal{C}} (b_c)_r \ .$$

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$$\sum_{r \in \mathcal{R}} y_r (f_c(y))_r \ \leq \ \sigma \inf_{b \in \mathcal{B}_c} \sum_{r \in \mathcal{R}} y_r b_r$$

for all price vectors $y \in \mathbb{R}^{\mathcal{R}}_{\geq 0}$ and some fixed $\sigma \geq 1$ (a σ-approximate **oracle**).

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for all price vectors $y \in \mathbb{R}^{\mathcal{R}}_{\geq 0}$ and some fixed $\sigma \geq 1$ (a σ-approximate **oracle**).

Task

- Find a $b_c \in \mathrm{conv}(\mathcal{B}_c)$ for each $c \in \mathcal{C}$ with minimum **congestion**

$$\max_{r \in \mathcal{R}} \sum_{c \in \mathcal{C}} (b_c)_r \,.$$

Resource sharing for global routing

Each net is a customer.

Define a resource r (with a capacity $\mathrm{cap}(r)$) for

- each edge of the global routing graph
- the total power consumption
- the critical area (for estimating the yield loss)
- each edge of the timing propagation graph

For each net c, let $\mathcal{B}_c$ contain, for each routing solution s for c, the vector $\left(\frac{\mathrm{usg}(s,r)}{\mathrm{cap}(r)}\right)_{r\in\mathcal{R}}$ where $\mathrm{usg}(s,r)$ tells how much s uses from r.

- We look for a solution with congestion at most 1.
- An objective function can be viewed as an additional resource (determine its capacity by binary search).

Rounding

- A solution to the min-max resource sharing instance is a convex combination $\sum_{b\in\mathcal{B}_c} p_b b$ of solutions for each net c ($p_b \geq 0$ ($b \in \mathcal{B}_c$), $\sum_{b\in\mathcal{B}_c} p_b = 1$)
- However, we need a single solution for each net.
- Use randomized rounding: randomly choose b with probability p_b, independently for each net. (Raghavan, Thompson [1987,1991], Raghavan [1988])
- If no solution consumes very much of any capacity, this yields a solution that exceeds capacities only slightly.
- New rounding and correction algorithms (e.g., Harris, Srinivasan [2013]) not better in practice (Bihler [2017])

Algorithms for min-max resource sharing

	oracle	# oracle calls
Grigoriadis, Khachiyan [1994]	strong, bounded	$\tilde{O}(\omega^{-2}\|\mathcal{C}\|^2)$
Grigoriadis, Khachiyan [1994]	strong, bounded	$\tilde{O}(\omega^{-2}\|\mathcal{C}\|)$ random.
Grigoriadis, Khachiyan [1996]	strong, unbounded	$\tilde{O}(\omega^{-2}\|\mathcal{C}\|\|\mathcal{R}\|)$
Jansen, Zhang [2008]	weak, unbounded	$\tilde{O}(\omega^{-2}\|\mathcal{C}\|\|\mathcal{R}\|)$
Müller, Radke, Vygen [2011]	weak, unbounded	$\tilde{O}(\omega^{-2}(\|\mathcal{C}\| + \|\mathcal{R}\|))$
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All these algorithms and predecessors for special cases (Plotkin, Shmoys, Tardos [1995], Young [1995], Villavicencio, Grigoriadis [1996], Garg, Könemann [2007], ...) compute a $\sigma(1+\omega)$-approximate solution for any given $\omega > 0$.

Running times dominated by number of oracle calls.
Logarithmic terms and dependency on σ omitted.

For a strong oracle, σ can be chosen arbitrarily close to 1.

A bounded oracle respects a given upper bound on resource usage.

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Weak duality

Let
$$\lambda^* := \inf\left\{\max_{r\in\mathcal{R}} \sum_{c\in\mathcal{C}} (b_c)_r : b_c \in \mathcal{B}_c \,(c \in \mathcal{C})\right\}$$
(the "*optimum congestion*").

Lemma (Weak duality)

Let $y \in \mathbb{R}^{\mathcal{R}}_{\geq 0}$ be some price vector, not all-zero, and $opt_c(y) := \inf_{b\in\mathcal{B}_c} \sum_{r\in\mathcal{R}} y_r b_r$. Then

$$\frac{\sum_{c\in\mathcal{C}} opt_c(y)}{\sum_{r\in\mathcal{R}} y_r} \leq \lambda^*.$$

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Proof

Let $(b_c \in \mathcal{B}_c)_{c\in\mathcal{C}}$ be a solution with congestion $\leq (1+\delta)\lambda^*$. Then

$$\frac{\sum_{c\in\mathcal{C}} \mathrm{opt}_c(y)}{\sum_{r\in\mathcal{R}} y_r} \leq \frac{\sum_{c\in\mathcal{C}} \sum_{r\in\mathcal{R}} y_r (b_c)_r}{\sum_{r\in\mathcal{R}} y_r} = \frac{\sum_{r\in\mathcal{R}} y_r \sum_{c\in\mathcal{C}} (b_c)_r}{\sum_{r\in\mathcal{R}} y_r}$$

$$\leq \frac{\sum_{r\in\mathcal{R}} y_r (1+\delta)\lambda^*}{\sum_{r\in\mathcal{R}} y_r} = (1+\delta)\lambda^*. \qquad \square$$

Bounding λ^*

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$$\frac{\sum_{c \in \mathcal{C}} opt_c(y)}{\sum_{r \in \mathcal{R}} y_r} \leq \lambda^*.$$

Corollary

Let $b_c := f_c(\mathbb{1})$ ($c \in \mathcal{C}$) and $\lambda^{ub} := \max_{r \in \mathcal{R}} \sum_{c \in \mathcal{C}} (b_c)_r$. Then

$$\frac{\lambda^{ub}}{|\mathcal{R}|\sigma} \leq \frac{\sum_{r \in \mathcal{R}} \sum_{c \in \mathcal{C}} (b_c)_r}{|\mathcal{R}|\sigma} \leq \frac{\sum_{c \in \mathcal{C}} opt_c(\mathbb{1})}{|\mathcal{R}|} \leq \lambda^* \leq \lambda^{ub} \, .$$

□

Scaling

We know $\frac{\lambda^{ub}}{|\mathcal{R}|\sigma} \leq \lambda^* \leq \lambda^{ub}$.

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We know $\frac{\lambda^{ub}}{|\mathcal{R}|\sigma} \leq \lambda^* \leq \lambda^{ub}$.

1. Set $j := 0$.
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4. If $\lambda^{(j)} \leq \frac{1}{2}$, then increment j and go to 2.
5. Now $\frac{1}{5\sigma} \leq \lambda^{*(j)} \leq 1$.
6. Find a solution with congestion $\lambda^{(j)} \leq \sigma(1 + \frac{\omega}{2})\lambda^{*(j)} + \frac{\omega}{10}$ (hence at most $\sigma(1 + \omega)$ times the optimum).

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Lemma (Main Lemma)

Let $\delta, \delta' > 0$. Suppose that $\lambda^* \leq 1$.
Then we can compute a solution with congestion at most

$$\sigma(1 + \delta)\lambda^* + \delta'$$

in

$$O\left((\delta\delta')^{-1}(|\mathcal{C}| + |\mathcal{R}|)\,\theta\,\log|\mathcal{R}|\right)$$

time, where θ is the time for an oracle call.

Core algorithm ("multiplicative weights method")

Input: An instance of the min-max resource sharing problem.
Output: A convex combination of vectors in $\mathcal{B}_c$ for each $c \in \mathcal{C}$.

$t := \left\lceil \frac{3\sigma \ln |\mathcal{R}|}{\delta\delta'} \right\rceil$, $\epsilon := \frac{\delta}{3\sigma}$.
$\alpha_r := 0$, $y_r := 1$ for each $r \in \mathcal{R}$.
$x_{c,b} := 0$ for each $c \in \mathcal{C}$ and $b \in \mathcal{B}_c$.
For $p := 1$ to t **do**: (perform t **phases**)
  For $c \in \mathcal{C}$ **do**:
    $b := f_c(y)$. (call oracle)

    $x_{c,b} := x_{c,b} + 1$. (record solution)
    $\alpha := \alpha + b$. (update resource consumption)
    For each $r \in \mathcal{R}$ with $b_r \neq 0$ **do**:
      $y_r := e^{\epsilon\alpha_r}$. (update prices)
$x_{c,b} := \frac{1}{t} x_{c,b}$ for each $c \in \mathcal{C}$ and $b \in \mathcal{B}_c$. (normalize)

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$\alpha_r := 0, \ y_r := 1$ for each $r \in \mathcal{R}$.
$x_{c,b} := 0, \ X_c := 0$ for each $c \in \mathcal{C}$ and $b \in \mathcal{B}_c$.
For $p := 1$ to t **do**: (perform t **phases**)
    While there exists $c \in \mathcal{C}$ with $X_c < p$ **do**:
        $b := f_c(y)$. (call oracle)
        $\xi := \min\{p - X_c, 1/\max\{b_r : r \in \mathcal{R}\}\}$.
        $x_{c,b} := x_{c,b} + \xi, \ X_c := X_c + \xi$. (record solution)
        $\alpha := \alpha + \xi b$. (update resource consumption)
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Proof of performance guarantee (sketch)

Lemma

Let (x, y) be the output of the algorithm with congestion $\lambda := \max_{r \in \mathcal{R}} \lambda_r$, where

$$\lambda_r := \sum_{c \in \mathcal{C}} \Big(\sum_{b \in \mathcal{B}_c} x_{c,b} b \Big)_r$$

Then

$$\lambda \leq \frac{1}{\epsilon t} \ln \Big(\sum_{r \in \mathcal{R}} y_r \Big).$$

Proof of performance guarantee (sketch)

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Then

$$\lambda \leq \frac{1}{\epsilon t} \ln \Big(\sum_{r\in\mathcal{R}} y_r \Big).$$

Proof: For $r \in \mathcal{R}$:

$$\lambda_r = \sum_{c\in\mathcal{C}} \sum_{b\in\mathcal{B}_c} x_{c,b} b_r = \frac{\alpha_r}{t} = \frac{1}{\epsilon t} \ln \left(e^{\epsilon\alpha_r} \right) = \frac{1}{\epsilon t} \ln y_r \leq \frac{1}{\epsilon t} \ln \Big(\sum_{r\in\mathcal{R}} y_r \Big).$$

$\square$

Proof of performance guarantee (sketch)

Lemma (Main Lemma)

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- Initially, we have $\Big(\sum_{r \in \mathcal{R}} y_r^{(0)}\Big) = |\mathcal{R}|$.
- Price y_r multiplied by $e^{\epsilon \xi b_r} \leq (1 + (e^\epsilon - 1)\xi b_r)$ in each iteration.
- Short calculation yields

$$\sum_{r \in \mathcal{R}} y_r^{(p)} \leq \sum_{r \in \mathcal{R}} y_r^{(p-1)} + \epsilon' \sum_{c \in \mathcal{C}} \mathrm{opt}_c(y^{(p)}),$$

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Together with $\lambda \leq \frac{1}{\epsilon t} \ln\left(\sum_{r \in \mathcal{R}} y_r^{(t)}\right)$, this proves the claim. $\square$

Number of oracle calls

After each oracle call

- either $X_c = p$ (happens only $t\,|\mathcal{C}|$ times)
- or $\xi b_r = 1$ for some $r \in \mathcal{R}$ (increases the price of r by e^{ϵ}).

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The above scaling algorithm computes a $\sigma(1+\omega)$-approximate solution in

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time, where θ is the time for an oracle call.

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Extensions for practical application:

- Most oracle calls not necessary; reuse previous result if still good enough. Use lower bounds to decide
- Avoid binary search by re-scaling objective function resource
- Parallelization (without loss of theoretical guarantees!)

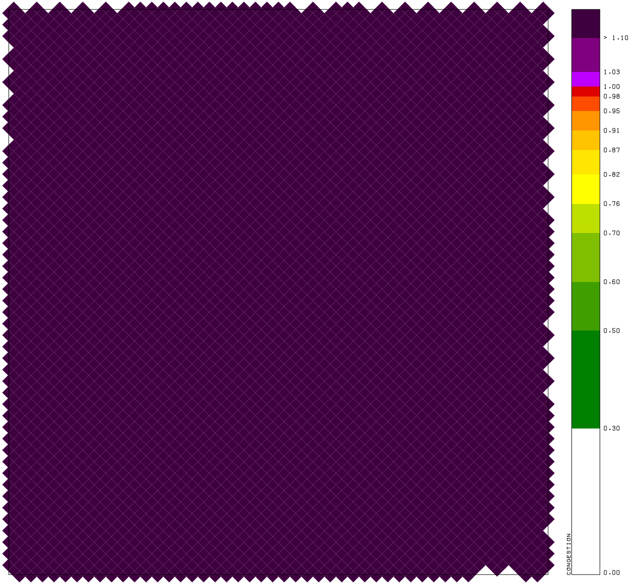
The algorithm in practice

- In practice, results are much better than theoretical performance guarantees. Usually 20–100 iterations suffice.
- Only few constraints are violated (even after rounding); these are corrected easily by *ripup-and-reroute*.
- Detailed routing can realize the solution well, due to good capacity estimations.
- Small integrality gap and approximate dual solution
$\Rightarrow$ infeasibility proof can be found for most infeasible instances
- New constraints can be added easily

Open problems:

- What if one resource is very bad? Guarantee for the rest?
- Warmstart with good prices?
- Need binary search?

The algorithm in practice



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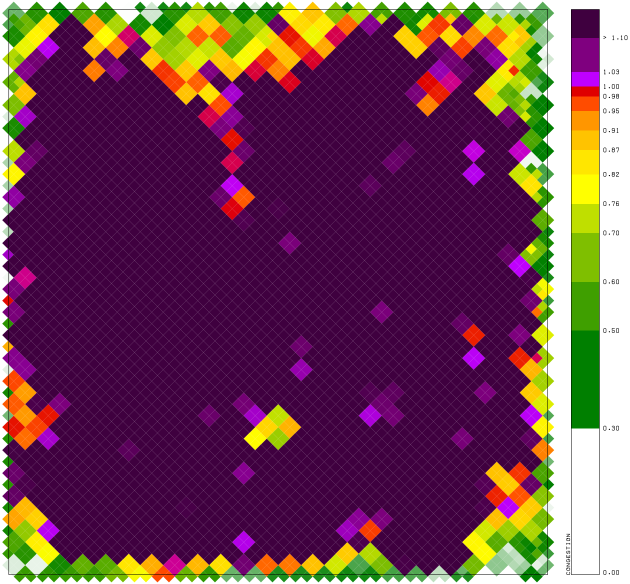

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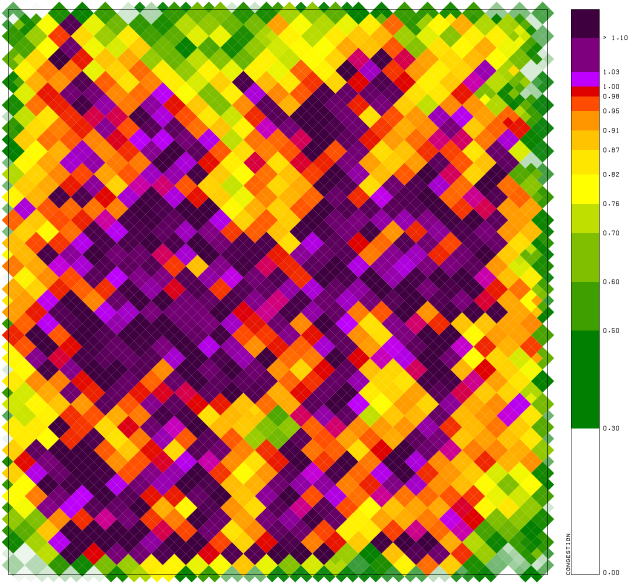

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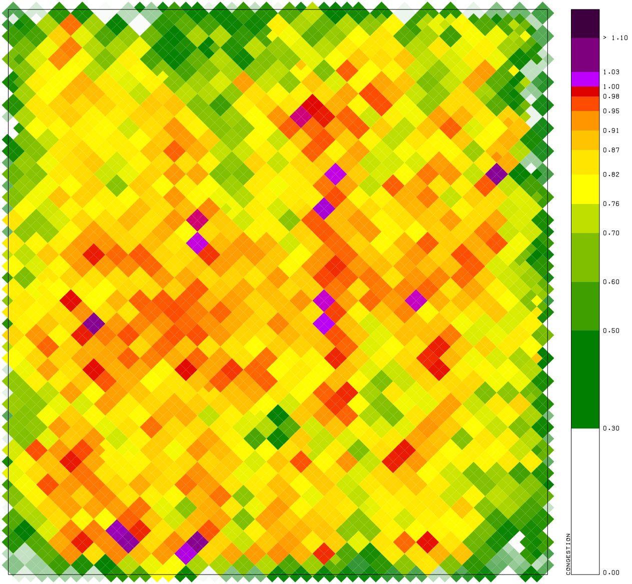

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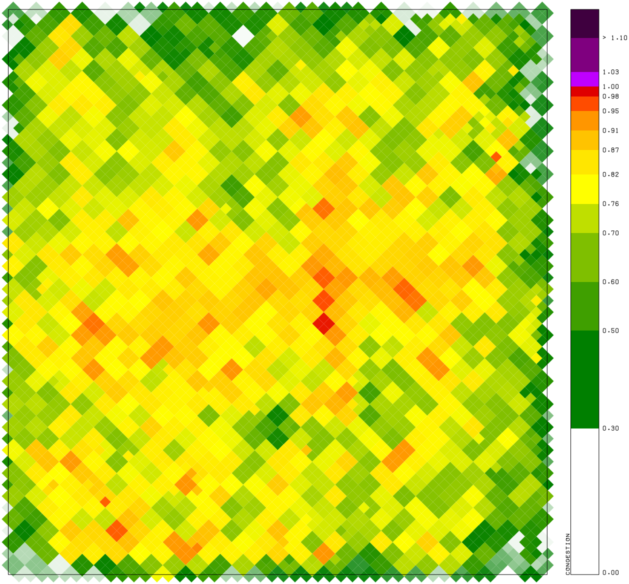

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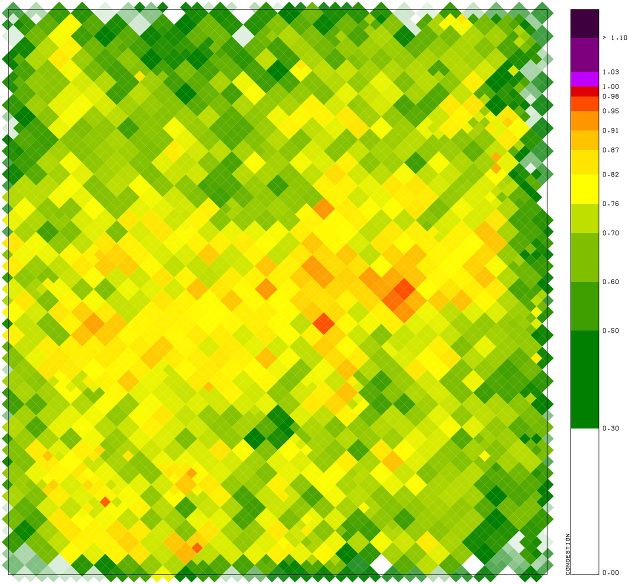

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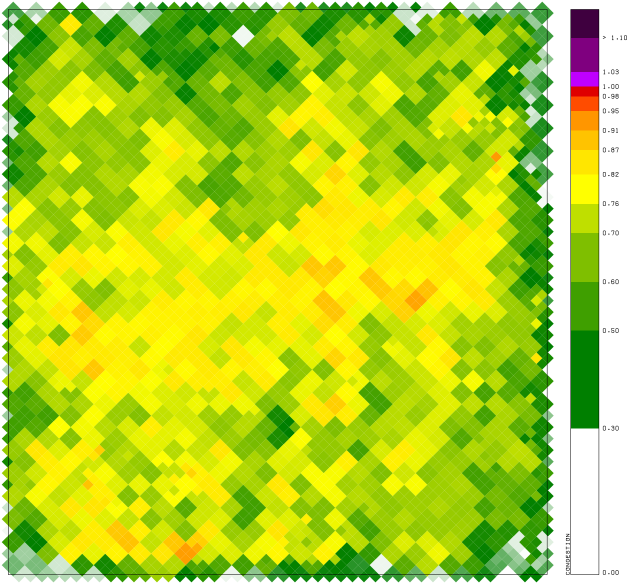

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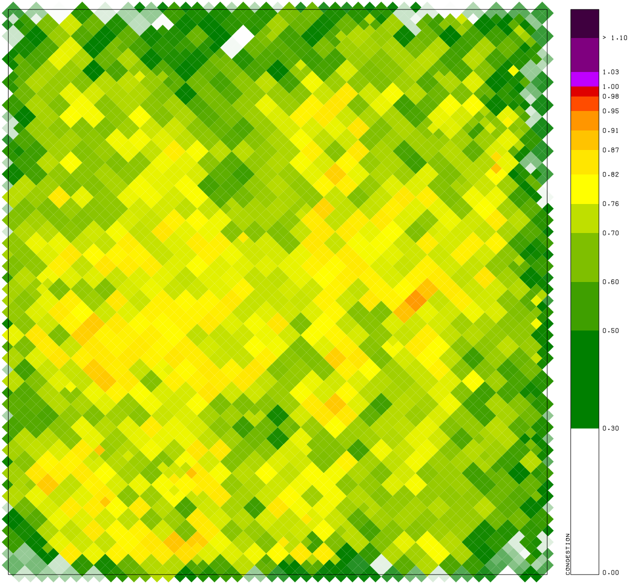

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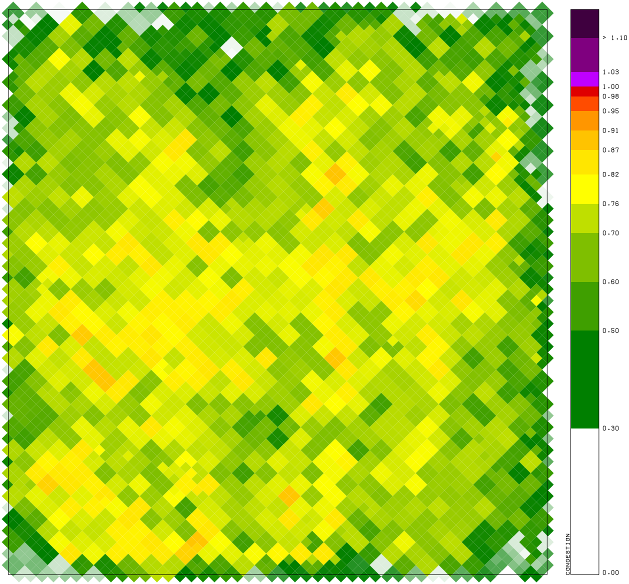

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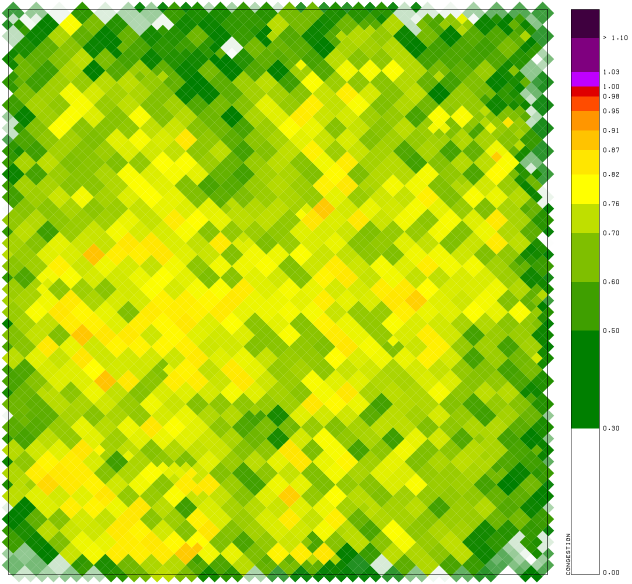

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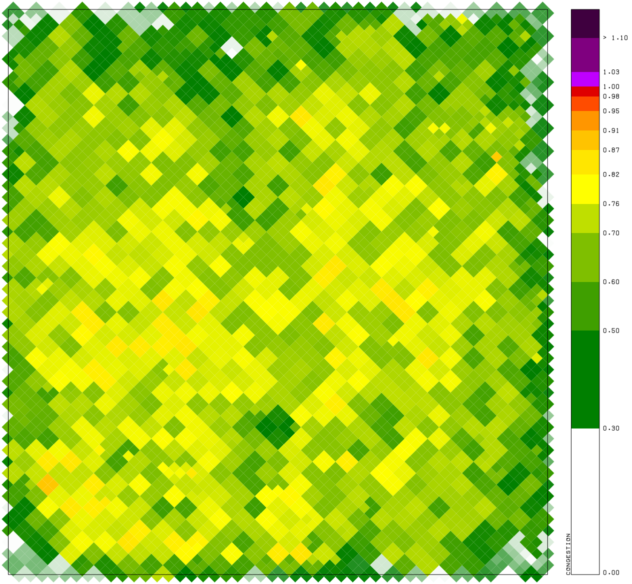

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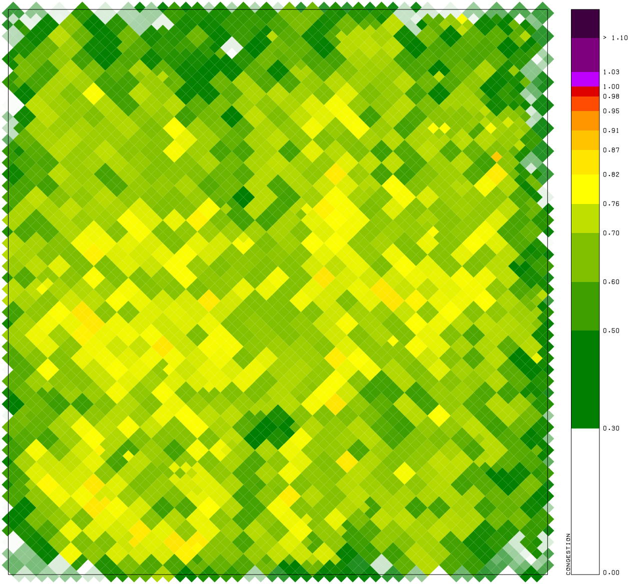
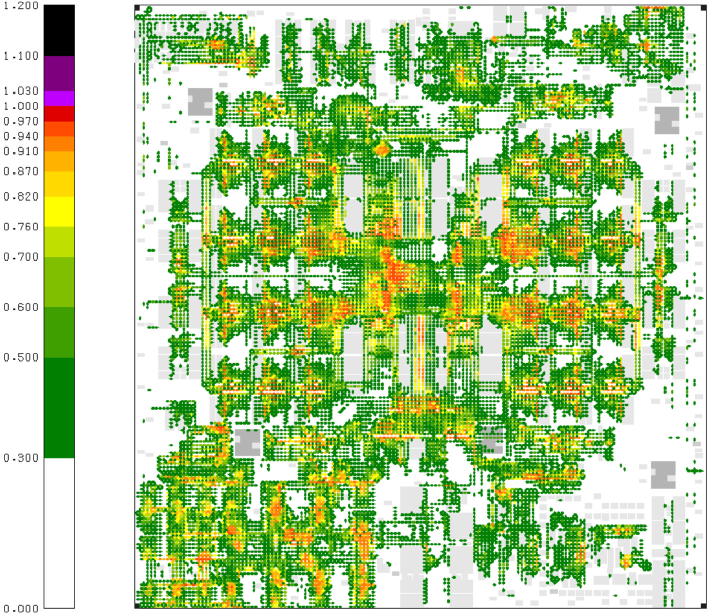


Near optimality

chip (# nets)	$\epsilon = \frac{100}{t}$	λ_{fract}	λ_{lb}	λ_{rounded}	λ_{final}	**% gap (fract.)**	**% gap (final)**	**running time**
Rose	0.5	0.950	0.941	2.000	1.000	1.91	1.96	0:01:45
(594 084)	1.0	0.950	0.939	2.000	1.000	1.97	2.01	0:01:08
	5.0	0.949	0.930	2.000	1.000	2.41	2.43	0:00:28
Georg	0.5	0.950	0.942	2.000	1.000	1.53	1.56	0:02:38
(783 685)	1.0	0.950	0.940	3.000	1.000	1.61	1.63	0:01:40
	5.0	0.949	0.934	2.000	1.000	1.90	1.91	0:00:49
Camilla	0.5	0.950	0.937	4.000	1.000	2.52	2.56	0:51:09
(3 582 559)	1.0	0.950	0.933	4.000	1.000	2.59	2.61	0:33:05
	5.0	0.950	0.918	3.000	1.000	3.34	3.33	0:11:23
Tomoko	0.5	1.629	1.449	3.000	1.600	1.12	1.14	0:23:43
(5 340 088)	1.0	1.625	1.449	3.000	1.600	1.18	1.20	0:15:21
	5.0	1.667	1.369	2.500	1.600	1.36	1.36	0:07:09
Andre	0.5	0.950	0.938	3.000	1.000	2.25	2.38	1:18:54
(7 039 094)	1.0	0.950	0.935	3.000	1.000	2.38	2.48	0:50:43
	5.0	0.950	0.923	3.000	1.000	2.92	2.96	0:17:32

Global routing result for the telecommunication chip

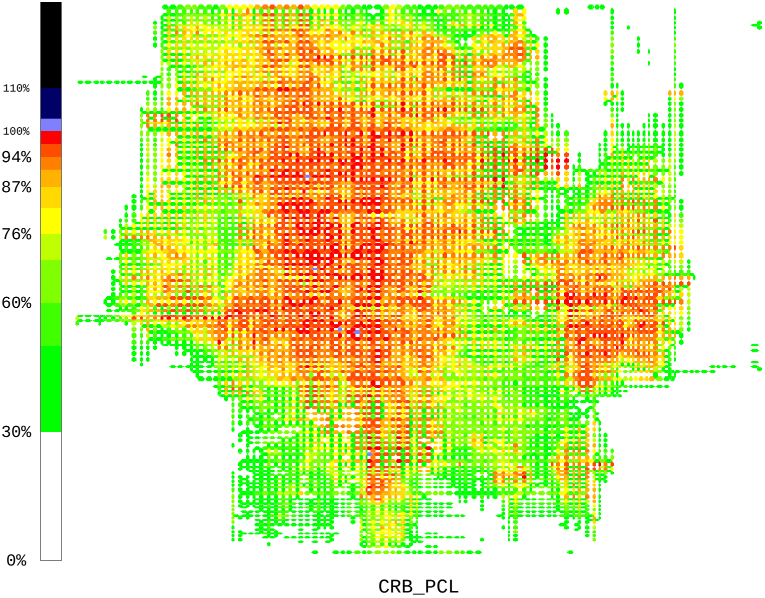


Overall result: comparison to an industrial router

	Time (h:mm:ss)		Wires	Vias	Scenic nets			Errors
	BonnRoute	Total	(m)		25%	50%	100%	
Industrial		9:18:14	16.57	11 160 081	44 419	21 841	2807	570
Bonn	1:17:35	3:55:32	14.86	8 640 867	2689	355	31	222
		-57.81%	-10.3%	-22.6%	-93.9%	-98.4%	-98.9%	-61.1%

- 14 nm testbed (14 instances with a total of 1 159 326 nets)
- both routers with 20 threads
- BonnRoute followed by industrial cleanup tool

Congestion map of a difficult instance



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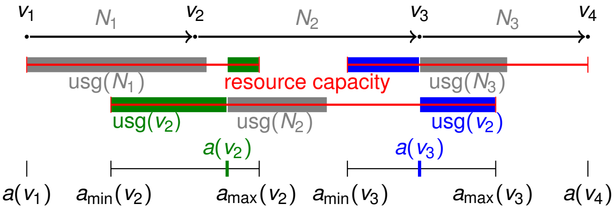
Modeling timing constraints via dynamic delay budgets

- Timing constraints are modeled by an acyclic digraph
- Arrival times at sources and latest allowed arrival times at sinks are given
- Delays on arcs depend on routing solution
- Add a resource for each of these arcs
- Add new customers for determining arrival times at intermediate vertices
- Results in dynamic delay budgets



Theorem (Held, Müller, Rotter, Scheifele, Traub, Vygen [2018])

Let $\omega > 0$.

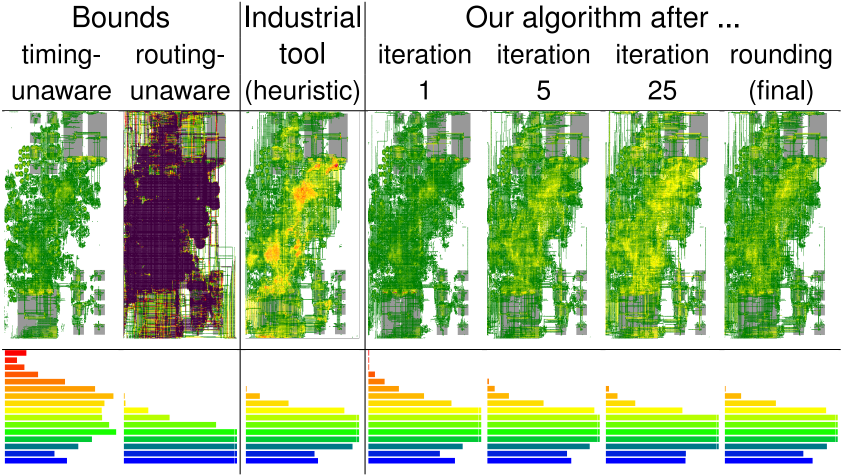
(a) *Given a* σ*-approximate routing oracle, we can compute, with* $\tilde{O}(\omega^{-2}(\mathcal{C}| + |\mathcal{R}|))$ *oracle calls, a solution that minimizes the congestion up to a factor* $\sigma(1 + \omega)$*.*

(b) *For nets with a bounded number of pins, we can obtain* $\sigma = 1$ *in polynomial time. Then, if there is a solution that satisfies all constraints, we get a solution that overloads edges by at most a factor* $1 + \omega$ *and violates timing constraints by at most* $\omega \cdot h$*, where* h *is a constant that depends on the instance only.*

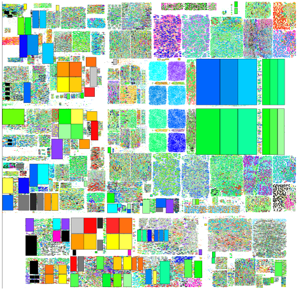
Sketch of proof:

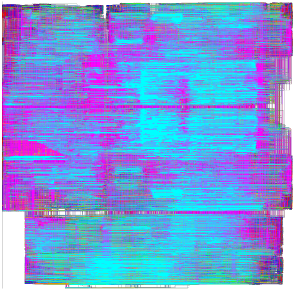
- Oracle for arrival time customers is trivial
- Resource sharing algorithm yields (a)
- Routing oracle for routing a net based on Dijkstra-Steiner algorithm (Hougardy, Silvanus, Vygen [2017])
- Increase capacities of timing resources so that all source-sink paths have the same capacity h.

Experimental results

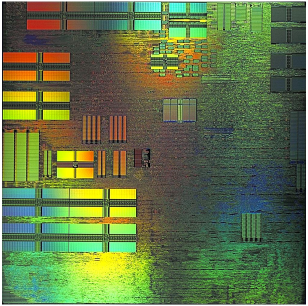


The world's fastest computer (Summit, 2018)





Conclusion: mathematics leads to better chips!



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