This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

Perfect graphs with unique P_4-structure

Stefan Hougardy [1]
Humboldt-Universität zu Berlin, Germany

October 1995, revised June 1996

Abstract. We will extend Reed's Semi-Strong Perfect Graph Theorem by proving that unbreakable C_5-free graphs different from a C_6 and its complement have unique P_4-structure.

Introduction

A graph is called *perfect* if for all of its induced subgraphs the chromatic number and the clique-number are the same. The notion of perfect graphs was introduced by Berge in 1960 [1] who also proposed two characterizations of perfect graphs. The first one is the famous Strong Perfect Graph Conjecture which states that a graph is perfect if and only if it contains no cycle of length at least five or a complement of such a cycle as an induced subgraph. This conjecture is open up to today. A second characterization conjectured by Berge was proved by Lovász [8] in 1972 and states that a graph is perfect if and only if its complement is perfect. This result is known as the Perfect Graph Theorem.

One of the most outstanding open problems in algorithmic graph theory is to determine the complexity of recognizing perfect graphs. Results of Lovász [8], Padberg [10] and Bland et al. [2] imply, as it was first observed by Cameron [3] in 1982, that the problem of recognizing perfect graphs is in co-$\mathcal{NP}$. So far it is not known whether

[1] Humboldt-Universität zu Berlin, Institut für Informatik, Lehrstuhl Algorithmen und Komplexität, Unter den Linden 6, 10099 Berlin, Germany, hougardy@informatik.hu-berlin.de

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

this problem also belongs to $\mathcal{NP}$, i.e., we do not know of any reasonable way to certify the perfection of an arbitrary graph.

One weak form of such a certificate is obtained via the Perfect Graph Theorem: to prove the perfection of a graph it is enough to show that its complement is perfect. In attempting to generalize this kind of certificate, Chvátal [4] invented in 1984 the notion of *P_4-structure*. For a given graph, its P_4-structure is defined as the 4-uniform hypergraph on the same vertex set as the original graph whose edges are all the 4-element sets that induce a P_4 (i.e., a path on four vertices) in the original graph. We say that a graph G has *unique P_4-structure* if any other graph that has isomorphic (as a hypergraph) P_4-structure to G is isomorphic to G or to the complement of G. A graph G has *strongly unique P_4-structure* if any other graph that has the same P_4-structure as G is equal to G or $\overline{G}$. The C_5 is an example of a graph that has unique but not strongly unique P_4-structure.

Chvátal [4] conjectured that the perfection of a graph depends solely on its P_4-structure. He was led to this conjecture by observing that odd cycles and their complements have unique P_4-structure. Therefore the truth of the Strong Perfect Graph Conjecture would imply his conjecture. Moreover, as the P_4 is a self-complementary graph, the P_4-structure of a graph and its complement are isomorphic. This shows that Chvátal's conjecture implies the Perfect Graph Theorem. Chvátal therefore suggested his conjecture be called the *Semi Strong Perfect Graph Conjecture*. In 1987, Reed [12] proved the conjecture and so it is now known as the *Semi Strong Perfect Graph Theorem*.

Chvátal has shown that to prove the Strong Perfect Graph Conjecture it is enough to have it proved for the class of so called *unbreakable* graphs. We will prove as a main result in Section 4 that C_5-free unbreakable graphs different from C_6 and its complement have unique P_4-structure. This result shows that the Semi Strong Perfect Graph Theorem and the Perfect Graph Theorem are equivalent for the class of C_5-free unbreakable graphs.

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

1 Notations

If x and y are adjacent vertices in a graph, we say that x *sees* y. Otherwise we say that x *misses* y. For a set S of vertices we say that z *disagrees* on S if it sees neither all nor none of the vertices of S. Otherwise we say that z is *homogeneous* on S. We use the notation xy as a short form for the edge $\{x, y\}$. The vertices x and y are called the *endpoints* of the edge xy. We denote the complement of a graph G by $\overline{G}$. All subgraphs in this paper are *induced* subgraphs.

A graph is called *perfect* if for all of its induced subgraphs the chromatic number and the clique number are the same. If a graph is not perfect then it is called *imperfect.* An imperfect graph that has the property that all its proper induced subgraphs are perfect is called *minimal imperfect.*

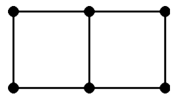
The *neighborhood* of a vertex x is denoted by $N(x)$. Sometimes we will also write $N_G(x)$ to make clear that x is a vertex of the graph G.

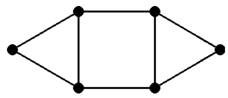
A path or cycle on k vertices is denoted by P_k respectively C_k. For simplicity of notation we will often denote a path or cycle by just listing its vertices, e.g., $abcd$ may stand for the path on four vertices $\{a, b, c, d\}$ and edges ab, bc, cd. For a path $x_1x_2\dots x_k$ the vertices x_1 and x_k are called the *endpoints* of the path. We also say that x_1 and x_k are *connected by the path* $x_1x_2\dots x_k$. All other vertices are called the *interior* of the path. For a P_4 the interior vertices are called the *midpoints.* A path or cycle is called *odd* or *even* if its length is odd or even. An *(odd) hole* is an (odd) induced cycle of length at least five. An *(odd) antihole* is the complement of an (odd) hole. A graph is called *Berge* if it contains neither an odd hole nor an odd antihole. A graph is called *disc* if it is a hole or an antihole.

A *hypergraph* $\mathcal{H}$ is a pair (V, F) where V is a finite set and F is a subset of the power set of V. The elements of V are called *vertices* and the elements of F are called *hyper-edges.* A hypergraph is called k*-uniform,* if all its hyper-edges have cardinality k. Two hypergraphs are *isomorphic* if there exists a bijection between their vertex sets that preserves all the hyper-edges.

A *domino* is the graph on vertices a, b, c, d, e, f with edges $ab, bc, cd, de, ef, fa, be$. The graph $\mathcal{F}$ is the complement of a domino, i.e., the graph on vertices a, b, c, d, e, f

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430





domino $\mathcal{F}$

Figure 1: Some special graphs.

and edges $ab, bc, cd, da, ea, ed, fb, fc$.

We denote the end of a proof by $\Box$ and the end of a proof of a claim within a proof by $\Diamond$.

2 Known results on perfect graphs

One of the most important results we will make use of in this paper is the Perfect Graph Theorem due to Lovász [9]. It states that

a graph is perfect if and only if its complement is perfect.

A *star-cutset* C in a graph G is a set of vertices such that $G - C$ is disconnected and there exists some vertex v in C that is adjacent to all other vertices in C. The vertex v is called a *center* of the star-cutset. A graph is called *unbreakable* if neither the graph nor its complement contains a star-cutset. Chvátal [5] proved that

minimal imperfect graphs are unbreakable.

Let S be a proper subset of the vertex set of a graph G. Then the vertices in $G - S$ can be partitioned into three classes: vertices that have no neighbor in S are called S-*null*; vertices that are adjacent to every vertex in S are called S-*universal*; all other vertices are called S-*partial*. Using this terminology a proper subset H of a graph G is called a *homogeneous set* if $|H| \geq 2$ and no vertex in $G - H$ is H-partial. Lovász [9] proved that

no minimal imperfect graph contains a homogeneous set.

A graph is called *weakly triangulated* if neither the graph nor its complement contains an induced cycle of length greater than four. Hayward [7] proved that

weakly triangulated graphs are perfect.

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

An *endomorphism* of a graph $G = (V, E)$ is a mapping $f : V \to V$ such that for any edge xy in G the image $f(x)f(y)$ is an edge in G. The endomorphism is *proper* if $f(V)$ is a proper subset of V. It was shown by Reed [11] that

no minimal imperfect graph admits a proper endomorphism.

3 Reed's Semi Strong Perfect Graph Theorem

To prove the Semi Strong Perfect Graph Theorem Reed actually proved the following theorem.

Theorem 1 (Reed [12]) *Let G and H be P_4-isomorphic graphs such that G is neither H nor $\overline{H}$. Then at least one of the following holds:*

(i) H contains a proper induced subgraph isomorphic to C_5.

(ii) H or $\overline{H}$ has a star-cutset.

(iii) H or $\overline{H}$ has a proper endomorphism.

The Semi Strong Perfect Graph Theorem now follows immediately from this result, as no minimal imperfect graph satisfies any of these conditions. Reed's proof of Theorem 1 relied on yet two other theorems and one lemma which we will state next.

Theorem 2 (Reed [12]) *If G and H are P_4-isomorphic graphs which are invariant on some disc of size at least six then either*

(i) $G = H$, or

(ii) H or $\overline{H}$ has a star-cutset, or

(iii) H contains a C_5 as a proper subgraph. □

Theorem 3 (Reed [12]) *Consider an unbreakable graph H, containing no C_5, that is P_4-isomorphic to a graph G. If some set D induces a C_6 in H and an $\mathcal{F}$ (see Figure 1) in G then H has a proper endomorphism.* □

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

Lemma 1 (Chvátal[4], Hayward [6]) *Discs have unique P_4-structure. The only exception are discs of size six that have the same P_4-structure as the graph $\mathcal{F}$ respectively $\overline{\mathcal{F}}$ (see Figure 2). Discs of size ≥ 7 have strongly unique P_4-structure.* □

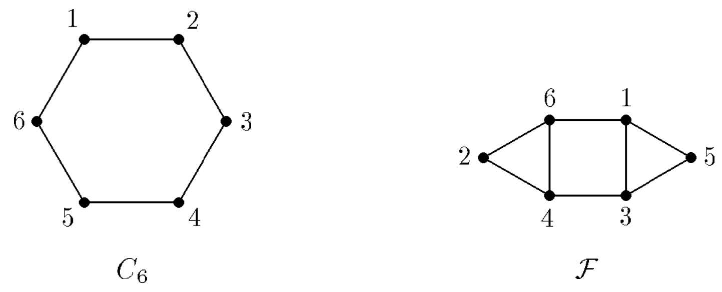


Figure 2: Two graphs with the same P_4-structure.

Using Theorems 2 and 3 Reed proved Theorem 1 as follows: Let G and H be P_4-isomorphic graphs such that G is neither H nor $\overline{H}$. We only have to show that H fulfills at least one of the three conditions in Theorem 1. Hayward [7] proved that no weakly triangulated graph on at least 3 vertices is unbreakable. Thus if H is weakly triangulated or contains a C_5 then condition (ii) or (i) is satisfied. Hence we may assume that H contains a disc of size at least six. Let D be the set of vertices of this disc inducing the subgraphs D_G and D_H of G and H, respectively. If $D_H = D_G$ then (i) or (ii) holds by Theorem 2. If $D_H = D_{\overline{G}}$ then (i) or (ii) holds by Theorem 2 with $\overline{G}$ in place of G. Thus we can assume $D_H \neq D_G$ and $D_H \neq D_{\overline{G}}$. Now from Lemma 1 we know that D_G or $D_{\overline{G}}$ must be the graph $\mathcal{F}$. If $D_G = \mathcal{F}$ then $(i), (ii)$ or (iii) holds by Theorem 3; if $D_{\overline{G}} = \mathcal{F}$ then $(i), (ii)$ or (iii) holds by Theorem 3 with $\overline{G}$ in place of G.

4 Graphs with unique P_4-structure

Theorem 1 of Reed says that if a C_5-free unbreakable graph has the property that neither the graph nor the complement has a proper endomorphism then the graph has unique P_4-structure. In this section we will prove a generalization of this result.

Reed himself suggested [12] that the condition that the graph is C_5-free might

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

be dropped from Theorem 1. However, in Theorem 2 the condition that H is C_5-free cannot be removed. Figure 3 shows two unbreakable graphs with the same P_4-structure that are invariant on a disc of size six but are not isomorphic.

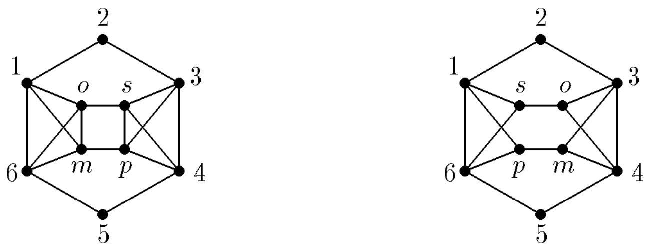


Figure 3: Two unbreakable graphs with the same P_4-structure

We will now prove that the condition on the proper endomorphism can be dropped from Theorem 1. The only exceptions are C_6 and its complement.

Theorem 4 *Let G and H be P_4-isomorphic graphs such that G is neither H nor $\overline{H}$. Then at least one of the following holds:*

(i) H contains a proper induced subgraph isomorphic to C_5.

(ii) H or $\overline{H}$ has a star-cutset.

(iii) H or $\overline{H}$ is a C_6.

A more compact equivalent formulation is given by the next theorem:

Theorem 5 *C_5-free unbreakable graphs different from C_6 and $\overline{C}_6$ have unique P_4-structure.* □

Note that this result implies that the Semi Strong Perfect Graph Theorem and the Perfect Graph Theorem are equivalent for the class of C_5-free unbreakable graphs.

The proof of the Semi Strong Perfect Graph Theorem that we sketched in Section 3 shows, that it is enough for a proof of Theorem 4 to demonstrate the truth of the following theorem which is an analogue of Theorem 3 of Reed.

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

Theorem 6 *Consider an unbreakable graph H, containing no C_5, that is P_4-isomorphic to a graph G. If some set D induces a C_6 in G and an $\mathcal{F}$ in H then G is a C_6.*

Proof. Let G be an unbreakable graph different from a C_6 that contains no C_5 and let H be a graph that is P_4-isomorphic to G. We may assume that G and H are defined on the same set of vertices such that four vertices induce a P_4 in G if and only if they induce a P_4 in H. Let D be a set of vertices that induces a C_6 in G and the graph $\mathcal{F}$ in H. We have to show that this leads to a contradiction. As G cannot contain a homogeneous set (otherwise G or $\overline{G}$ contains a star-cutset), there must exist a vertex a that is D-partial in G. A simple case analysis, using the fact that G is C_5-free and that D induces the graph $\mathcal{F}$ in H, shows that up to symmetry the graph induced in G by $D \cup \{a\}$ is of one of six types. Figures 4 and 5 show these six possibilities together with the corresponding graphs that are induced by $D \cup \{a\}$ in H. The dashed lines indicate edges where we do not care whether they exist or not.

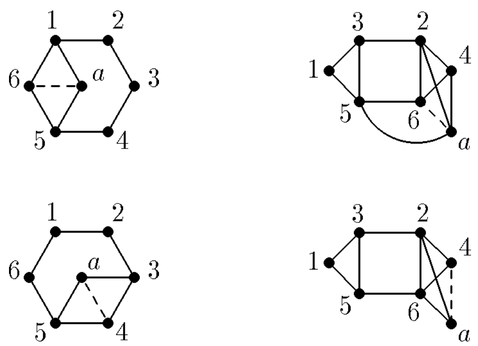


Figure 4: Possible types of partial vertices of the C_6: twins

For the following we assume that the vertices of D are labeled $1, \ldots, 6$ in the cyclic order they appear around the C_6 in G so that vertices 1 and 4 have degree 2 in F.

Thus, $V - D$ can be partitioned into sets

$$T_1, T_2, T_3, T_4, T_5, T_6, S_{\{1\}}, S_{\{4\}}, S_{\{1,3,5\}}, S_{\{1,3,4,5\}}, S_{\{2,4,6\}}, S_{\{1,2,4,6\}}, S_{\{2,3,5,6\}}, A, N$$

where:

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

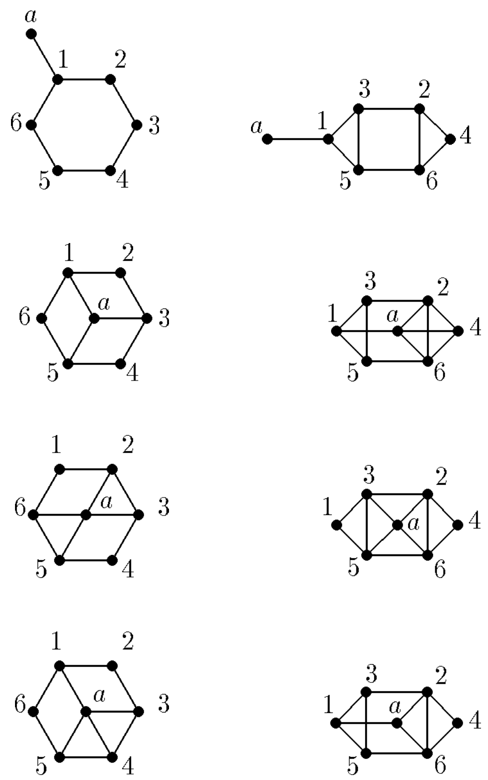


Figure 5: Possible types of partial vertices of the C_6: non-twins

(i) For each $i \in \{1, \ldots, 6\}$, $T_i = \{v | N_G(v) \cap D - i = N_G(i) \cap D - i\} = \{v | N_H(v) \cap D - i = N_H(i) \cap D - i\}$

(ii) For each $U \in \{\{1\}, \{4\}, \{1,3,5\}, \{1,3,4,5\}, \{2,4,6\}, \{1,2,4,6\}, \{2,3,5,6\}\}, S_U = \{v | N_G(v) \cap D = U\}$. Note that for each U and for each v in S_U, $N_H(v) \cap D$ is determined as shown in Figure 5.

(iii) A is the set of vertices adjacent in G to all of D, N is the set of vertices adjacent in G to none of D.

Claim 1 $T_2 = T_3 = T_5 = T_6 = \emptyset$.

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

Note that the same partitioning as stated above for $V - D$ exists for any other set D' which induces a C_6 in G and an F in H. In particular, for each v in some T_i, such a partition exists for $D_v = D - i + v$. It follows that no vertex w outside $T_2 + 2$ disagrees on two vertices of $T_2 + 2$ as otherwise w has an unallowable neighborhood either on D or on D_v for some v in T_2. Thus, T_2 is empty as otherwise $T_2 + 2$ is a homogeneous set. Similarly, T_3, T_5 and T_6 are also empty. $\diamond$

At this point, we decompose $V - D$ into $X_4 = T_4 \cup S_{\{1,3,5\}} \cup S_{\{1,3,4,5\}}$, $X_1 = T_1 \cup S_{\{2,4,6\}} \cup S_{\{1,2,4,6\}}$, S_1, S_4, S_{2356}, A, N. Note that for each vertex v in X_1, we have: $N_G(v) \cap \{2,3,5,6\} = \{2,6\}$, and $N_H(v) \cap \{2,3,5,6\} = \{3,5\}$, whilst for each vertex v of X_4 we have: $N_H(v) \cap \{2,3,5,6\} = \{2,6\}$, and $N_G(v) \cap \{2,3,5,6\} = \{3,5\}$.

Claim 2 *If* $S_1 \neq \emptyset$ *then* $S_{135} \cup S_{1345} = \emptyset$.

Let w be a vertex in S_1 and v be a vertex in $S_{135} \cup S_{1345}$. Assume that v and w are not adjacent in G. Then $w1v3$ induces a P_4 in G but not in H. Similarly if vw is an edge in G then it must also be an edge in H as otherwise $wv56$ is a P_4 in G but not in H. But now either $wv54$ is a P_4 in G but not in H or $wv64$ is a P_4 in H but not in G. $\diamond$

Claim 3 $S_1 = S_4 = \emptyset$.

Assume $S_1 \neq \emptyset$. Let P be a minimal path of $G - N(2) + 3$ from S_1 to $S_4 \cup X_4 \cup \{3,4,5,6\}$. Then, one endpoint w of P is in S_1 and all its interior vertices are in N, so the other endpoint v of P must be in $X_4 \cup S_4$. If v is in S_4 then $P + 1 + 2 + 3 + 4$ is a hole C (we can ofcourse assume G contains no hole of length seven or greater) so vw is an edge. But now C induces a C_6 in both H and G and again we are done.

Similarly v is not in T_4, as otherwise $P + 1 + 2 + 3$ is a hole of length at least seven or induces a C_6 in both G and H or it is a C_5 in G contradicting the fact that G is C_5-free.

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

Finally v is not in $S_{135} \cup S_{1345}$ by Claim 2. Thus P does not exist and so G is not unbreakable, a contradiction. By symmetry, S_4 is also empty. $\diamond$

Claim 4 $N = \emptyset$.

First note that there are no edges from a vertex $n \in N$ to a vertex $v \in S_{135} \cup S_{1345}$ as otherwise $nv12$ induces a P_4 in G but not in H. Similarly there are no edges from N to $S_{246} \cup S_{1246}$. Now there are no edges from a vertex $n \in N$ to a vertex $v \in T_1$ as otherwise $|N_G(n) \cap D_v| = 1$ contradicting Claim 3. Similarly, there are no edges from N to S_4. But now, $2 + N(2)$ is a cutset separating N from 4, a contradiction. $\diamond$

At this point, we have a partition of V into X_1, X_4, A, $S = S_{2356}$, and D.

Claim 5 *There is no edge between S and $X_1 \cup X_4$; there are all edges between A and $X_1 \cup X_4$.*

Let $s \in S$. If s sees a vertex $x \in T_4$ then $12sx$ is a P_4 in G but not in H. If s sees a vertex $x \in S_{\{1,3,5\}} \cup S_{\{1,3,4,5\}}$ then xs must be an edge in H as otherwise $1x2s$ is a P_4 in H but not in G. If x is in $S_{\{1,3,5\}}$ then $3sx4$ induces a P_4 in H but not in G. If x is in $S_{\{1,3,4,5\}}$ then $2sx4$ is a P_4 in G but not in H.

Let $a \in A$. If a misses a vertex $x \in T_4$ then $6a3x$ induces a P_4 in G therefore a must be D-universal in H and misses x. But then $1a2x$ is a P_4 in H but not in G. If a misses a vertex $x \in S_{\{1,3,5\}} \cup S_{\{1,3,4,5\}}$ then the P_4 $2a5x$ in G shows that a must also be D-universal in H and misses x in H. But now the set $\{1, a, 4, x\}$ induces a P_4 in exactly one of G and H. $\diamond$

Claim 6 $A = \emptyset$

If A is non-empty then S is also non-empty as otherwise $\overline{G}$ is disconnected. If there is no edge between A and S then for any vertex $a \in A$ we have that $a \cup N(a) - 4$

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

separates S from vertex 4. Thus there must exist a vertex s that has a neighbor $a \in A$. But then $\overline{N(s)}$ separates a from $\{2,3,5,6\}$ in $\overline{G}$. This contradicts the fact that G is unbreakable. So A is empty. $\diamond$

Claim 7 $S = \emptyset$

If $|S| \geq 2$ then S is a homogeneous set. If S is just one vertex s, then since G contains no C_5, there are no edges from $X_1 + 1$ to $X_4 + 4$ (or else a vertex from $X_1 + 1$ and one from $X_4 + 4$ together with $\{2, s, 5\}$ induce a C_5 in G. But now, $s + N(s)$ is a star cutset separating 1 from 4. So, S must be empty. $\diamond$

Now, if both X_1 and X_4 are empty then G is a C_6. So by symmetry, we can assume that X_1 is non-empty. If there are no two vertices x and y in $X_1 + 1$ with incomparable neighborhoods then let z be a vertex in $X_1 + 1$ with maximal neighborhood. Clearly $z + (N(z) - X_1 - 1)$ is a cutset separating $X_1 + 1 - z$ from 3. So, we can assume there are two vertices x and y in $X_1 + 1$ and two vertices w and z in $X_4 + 4$ such that $xz, yw \in E(G)$ and $xw, yz \notin E(G)$. Because G has no C_5 we see that either both or neither of xy, wz are edges of G. Note that xz and yw are also edges in H while xw and yz are not edges in H. Moreover xy and zw are edges in G if and only if they are non-edges in H. In either case, the sets $\{2, x, y, z, w, 5\}$ and $\{6, x, y, z, w, 3\}$ induce a C_6 in one of G or H and an F in the other with the vertices of degree 2 in these F in D. It follows from our previous remarks that a vertex v in $X_1 + 1 - x - y$ satisfies either $N(v) \cap \{x, y, z, w\} = \{x, y\}$, or $N(v) \cap \{x, y, z, w\} = \{z, w\}$. In either case, we easily arrive at a contradiction by considering the graph induced by $v, x, y, z, w, 2, 3, 5, 6$ as the set $\{v, x, y, z\}$ or $\{v, z, w, x\}$ induces a P_4 in exactly one of G and H. So, $X_1 + 1 = \{x, y\}$. By symmetry, $X_4 + 4 = \{w, z\}$.

Thus G is either the *cube* or the graph Q as depicted in Figure 6 and H is easily seen to be isomorphic to G or $\overline{G}$. $\square$

The proof of Theorem 4 shows that except for the two graphs depicted in Figure 6, C_5-free unbreakable graphs even have *strongly* unique P_4-structure. Thus we have

Corollary 1 *C_5-free unbreakable graphs different from C_6, Q, cube and their comple-*

This paper appeared in: Discrete Mathematics 165/166 (1997), 421-430

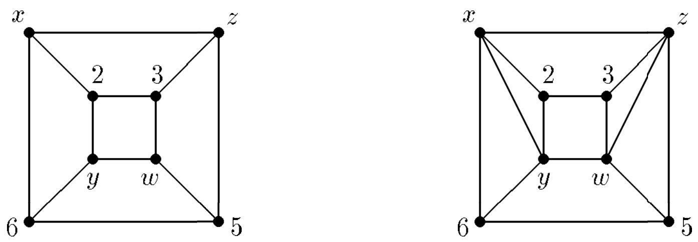


Figure 6: Two exceptional graphs: the *cube* and the graph Q.

ments have strongly unique P_4-structure. □

Acknowledgement

I am very grateful to an anonymous referee for simplifying my original proof of Theorem 4.

References

[1] C.Berge, *Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind,* Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe (1961), 114–115

[2] R.G.Bland, H.-C. Huang, L.E.Trotter Jr., *Graphical properties related to minimal imperfection,* Ann. Discrete Math. 21 (1984), 181–192

[3] K.Cameron, *Polyhedral and algorithmic ramifications of antichains,* PhD Thesis, University of Waterloo (1982)

[4] V.Chvátal, *A Semi–Strong Perfect Graph Conjecture,* Ann. Discrete Math. 21 (1984), 279–280

[5] V.Chvátal, *Star–cutsets and perfect graphs,* J. Combin. Theory Ser. B 39 (1985), 189–199

[6] R.B.Hayward, *On the P_4-structure of perfect graphs. III. Weakly triangulated graphs,* McGill University SOCS, Technical Report 83.20, Montreal, Canada, 1983

[7] R.B.HAYWARD, *Weakly triangulated graphs,* J. Combin. Theory Ser. B 39 (1985), 200–209

[8] L.LOVÁSZ, *A characterization of perfect graphs,* J. Combin. Theory Ser. B 13 (1972), 95–98

[9] L.LOVÁSZ, *Normal hypergraphs and the Perfect Graph Conjecture,* Discrete Math. 2 (1972), 253–267

[10] M.W.PADBERG, *Perfect zero–one matrices,* Math. Programming 6 (1974), 180–196

[11] B.A.REED, *A Semi–Strong Perfect Graph Theorem,* PhD-Thesis, McGill University, Montreal 1986

[12] B.A.REED, *A Semi–Strong Perfect Graph Theorem,* J. Combin. Theory Ser. B 43 (1987), 223–240