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On the P_4-Structure of Perfect Graphs V. Overlap Graphs

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Abstract. Given a graph G we define its k-overlap graph as the graph whose vertices are the induced P_4's of G and two vertices in the overlap graph are adjacent if the corresponding P_4's in G have exactly k vertices in common. For $k = 1, 2, 3$ we prove that if the k-overlap graph of G is bipartite then G is perfect.

1 Introduction

A graph G is called *perfect* if for every induced subgraph H of G the chromatic number of H equals the clique number of H. The notion of perfect graphs was introduced by Berge [1]. In 1960 he posed the famous Strong Perfect Graph Conjecture which is still open:

Strong Perfect Graph Conjecture *A graph is perfect if and only if it does not contain an odd cycle of length at least five or its complement as an induced subgraph.*

An odd (resp. even) induced cycle of length at least five is called an odd (resp. even) *hole*. Graphs that contain neither odd holes nor complements of odd holes are called *Berge*. Using this terminology the Strong Perfect Graph Conjecture can be restated as: A graph is perfect if and only if it is Berge.

Together with the Strong Perfect Graph Conjecture Berge also made a weaker conjecture which has been proved by Lovász [17] in 1972 and is nowadays called the Perfect Graph Theorem.

Perfect Graph Theorem *The complement of a perfect graph is perfect.*

A P_4 is a path on four vertices. Two graphs G and H are called *P_4-isomorphic* if there exists an isomorphism between the vertices of G and H such that four vertices induce a P_4 in

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G if and only if their images under this isomorphism induce a P_4 in H. In 1984 Chvátal [3] conjectured that if a graph G is P_4-isomorphic to a perfect graph then G is perfect.

Chvátal [3] showed that this conjecture is implied by the Strong Perfect Graph Conjecture and it is easy to see that this conjecture implies the Perfect Graph Theorem. Therefore this conjecture has been called the Semi Strong Perfect Graph Conjecture. Reed [20] proved this conjecture in 1987 which is therefore nowadays called the Semi Strong Perfect Graph Theorem.

Semi Strong Perfect Graph Theorem *If a graph G is P_4-isomorphic to a perfect graph then G is perfect.*

The validity of the Semi Strong Perfect Graph Theorem shows that the perfectness of a graph depends solely on its P_4-structure. On the one hand this motivates to look for 'natural' decomposition schemes that are derived from the P_4-structure of a graph. Such approaches where made by Chvátal and Hoàng in [6] and [14] which where generalized by Chvátal in [5]. The most general approach of this kind was made by Chvátal, Lenhart and Sbihi in [7]. They proved that if the vertices of a graph G can be colored by two colors such that every induced P_4 of G is colored by one of certain possibilities, then G is perfect if and only if the two graphs induced by the vertices of G that received the same color, are perfect.

On the other hand the validity of the Semi Strong Perfect Graph Theorem suggests defining classes of perfect graphs solely in terms of the P_4-structure. This was for example done by Chvátal who conjectured that a graph G is perfect if its partner graph is bipartite (the *partner graph* of a graph G is the graph whose vertices are the vertices of G, and two vertices a and b in the partner graph are adjacent if there are vertices x, y, z in $G - \{a, b\}$ such that $\{a, x, y, z\}$ and $\{b, x, y, z\}$ each induce a P_4 in G). Hayward and Lenhart [13] proved that an even more general statement holds: If the partner graph of G is triangle free then G is perfect.

In this paper we will define several new classes of perfect graphs which can be derived from the P_4-structure. Given a graph G we define its *k-overlap graph* as the graph whose vertices are the induced P_4's of G and two vertices in the overlap graph are adjacent if the corresponding P_4's in G have exactly k vertices in common. We will prove for $k = 1, 2, 3$ that if the k-overlap graph of a Berge graph G is bipartite then G is perfect. (Actually we are proving some stronger statements). For $k = 3$ this generalizes results of Hayward and Lenhart on partner graphs [13].

The paper is organized as follows: The next section contains some basic definitions and auxiliary results needed in the later sections. Sections 3, 4 and 5 contain our results for the 3-, 2- and 1-overlap graphs. In Section 6 we compare our new classes of perfect graphs with the known classes.

2 Notation and auxiliary results

Given two vertices x and y in a graph G we say that x *sees* y if x and y are connected by an edge in G. If x does not see a vertex y then we say that x *misses* y. The *neighborhood* of a vertex x is defined as the set of vertices that are adjacent to x and it is denoted by $N(x)$.

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A path (resp. cycle) on k vertices is denoted as P_k (resp. C_k). For a path on four vertices we often will just list its set of vertices, e.g. $abcd$ stands for the path on vertices a, b, c and d with edges ab, bc and cd. We will denote cycles of length five and six in a similar way. An induced cycle of length at least five is called a *hole*. The complement of a hole is called an *antihole*.

Two vertices a and b of a graph G are called *partners* if there are vertices x, y, z in $G - \{a, b\}$ such that $\{a, x, y, z\}$ and $\{b, x, y, z\}$ each induce a P_4 in G. The *partner graph* of a graph G is the graph whose vertices are the vertices of G, and whose edges are the pairs of vertices that are partners in G.

A *star-cutset* C in a graph G is a set of vertices such that $G - C$ is disconnected and there exists some vertex v in C that is adjacent to all other vertices in C. Chvátal [4] proved that

no minimal imperfect graph contains a star-cutset.

Let S be a proper subset of the vertex set of a graph G. Then the vertices in $G - S$ can be partitioned into three classes: vertices that have no neighbor in S are called *S-null*; vertices that are adjacent to every vertex in S are called *S-universal*; all other vertices are called *S-partial*. Using this terminology a proper subset H of a graph G is called a *homogeneous set* if $|H| \geq 2$ and no vertex in $G - H$ is H-partial. Lovász [18] proved that

no minimal imperfect graph contains a homogeneous set.

A pair A, B of disjoint subsets of vertices of a graph G is called a *homogeneous pair* if i) there are at least two vertices in $G - A - B$; ii) at least one of the two sets A and B contains at least two elements and iii) no vertex in $G - A - B$ is A- or B-partial. Note that every graph on at least four vertices that contains a homogeneous set also contains a homogeneous pair. Chvátal and Sbihi [8] proved that

no minimal imperfect graph contains a homogeneous pair.

A vertex a is said to *dominate* a vertex b if $N(b) \subseteq N(a) \cup a$. Two vertices a and b are called a *comparable pair of vertices* if a dominates b or b dominates a. It is easy to see that

no minimal imperfect graph contains a comparable pair of vertices.

A graph is called *weakly triangulated* if neither the graph nor its complement contains an induced cycle of length greater than four. Hayward [11] proved that

weakly triangulated graphs are perfect.

A K_4 is a complete graph on four vertices. Tucker [22] proved that in a minimal imperfect Berge graph every vertex is contained in a K_4. Let $\omega(G)$ denote the clique number of a graph

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G and let $\alpha(G)$ denote its stability number. Tucker's result shows that the clique number of a minimal imperfect Berge graph must be at least four. By the Perfect Graph Theorem the same must hold for the stability number. Lovász [17] proved that a minimal imperfect graph G contains exactly $\omega(G) \cdot \alpha(G) + 1$ vertices. Therefore we know that

a minimal imperfect Berge graph has at least 17 vertices.

A recent result of Sebő [21] says that minimal imperfect graphs are $2\omega - 2$ (vertex-) connected. Together with the above mentioned result of Tucker this shows that

minimal imperfect Berge graphs are 6-connected.

3 3-overlap graphs

The main result of this section is the following theorem:

Theorem 1 *If the 3-overlap graph of a Berge graph G is triangle-free then G is perfect.*

The 3-overlap graph of an odd hole or an odd antihole is an odd hole. Therefore we obtain as a corollary that the perfectness of a graph G is already guaranteed if its 3-overlap graph is bipartite. It is worth to note that this latter property can obviously be checked in polynomial time.

Corollary 1 *If the 3-overlap graph of G is bipartite then G is perfect.*

To establish the correctness of Theorem 1 we will prove the following stronger result:

Theorem 2 *If G is Berge and does not contain any of the three graphs H_1, H_2 and H_3 or their complements as an induced subgraph then G is perfect.*

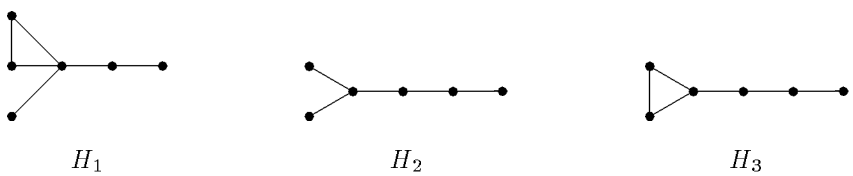

Figure 1: Forbidden subgraphs

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Proof. If neither G nor $\overline{G}$ contains an induced cycle of length at least six then G is weakly triangulated and therefore perfect. We thus may assume by symmetry that G contains an induced cycle C of length at least six.

Claim 1: *If $|C| > 6$ then either $G = C$ or C is a homogeneous set.*

To prove this assume that $G \neq C$ and C is not a homogeneous set. Then there must exist a vertex x not belonging to C that is partial on C. Let the vertices of C be labeled $a, b, c, d, \ldots$. Since x is partial on C we may assume that x sees c but does not see d. If x misses e and f then $fedcbx$ induces an H_2 or H_3 in G. Thus x must see at least one of e and f. If x sees f then it also must see e because otherwise G contains an induced C_5. Therefore x must see e. This shows that

$$x \text{ cannot have two consecutive non-neighbors on } C. \qquad (*)$$

If x sees neither b nor f then by $(*)$ xg must be an edge. But then $fgxcdb$ induces an H_2. Thus x must see at least one of f and b. By symmetry we may assume that x sees b.

Now if x does not see f then by $(*)$ it must see g. But then depending on the edge xa either $abxdef$ induces an H_2 or $abdegx$ induces an H_1. Therefore x must see f.

If x does not see g then by $(*)$ it must see h. But then $ghxecb$ induces an H_1. Therefore x must see g. By symmetry x must also be adjacent to a. But now $abxged$ induces an H_1.

Therefore the vertex x cannot exist and this proves Claim 1. $\diamond$

Claim 2: *If $|C| = 6$ then either $|G| \leq 13$ or G contains a homogeneous pair or a star-cutset.*

Let us assume that G has at least 14 vertices and that G contains no homogeneous set. We first consider all possible types of partial vertices for a C_6 such that no induced C_5 arises (see Figure 2).

A partial vertex of type 1 or type 3 cannot occur since otherwise the graph G contains H_2 as an induced subgraph. If there is a partial vertex of type 2 or type 4 then G contains H_3 as an induced subgraph. Thus the only possible types of partial vertices of a C_6 are 5, 6, 7, and 8.

If G contains only one C_6-partial vertex x then let A be the set of neighbors of x on the C_6 and let B be the remaining vertices of the C_6. Then the sets A and B form a homogeneous pair as soon as G contains at least 9 vertices. This shows that G contains at least two C_6-partial vertices.

Now it is easy to see that a vertex of type 5 and a vertex of type 7 cannot occur simultaneously in G. If two such vertices exist then they must be adjacent because otherwise G contains an induced C_5. But if these two vertices are adjacent then G contains $\overline{H}_3$.

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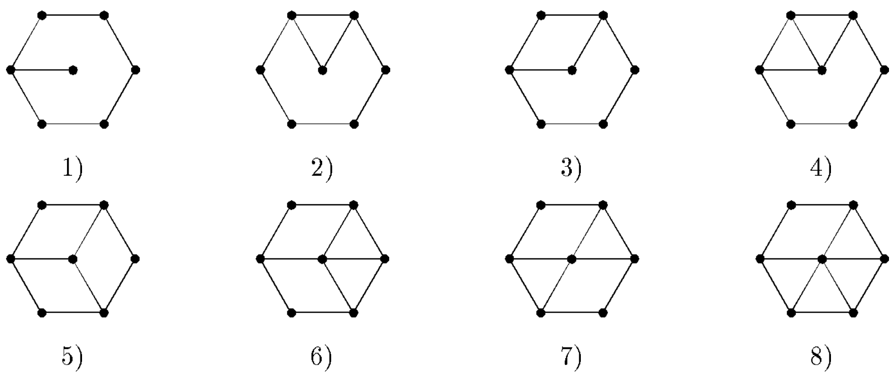

Figure 2: Possible types of partial vertices of a C_6

With a similar argument for the other combinations of two different partial vertices of types 5, 6, 7, and 8 one obtains that there is only one such possible combination. This is a combination of a partial vertex of type 6 with a partial vertex of type 8 that are adjacent and arranged as shown in Figure 3. In all other cases G would contain a C_5 or one of the graphs $\overline{H}_1$,$\overline{H}_2$ or $\overline{H}_3$ (In total there are 26 cases to check).

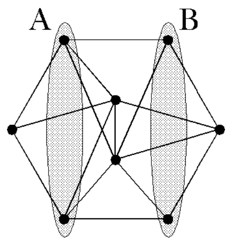

Figure 3: The only possible combination of two different partial vertices.

But in the case of Figure 3 the graph G would contain a homogeneous pair consisting of the sets A and B defined as shown in the figure.

This shows that all C_6-partial vertices must be of the same type.

If all partial vertices of C are of type 5 then let A and B be the disjoint sets consisting of every second vertex of C. Then since we assumed that G contains at least 14 vertices these two sets form a homogeneous pair.

Let us assume now that all partial vertices are of type 6. Then there cannot exist two such partial vertices that have the same neighbors on the C_6 because otherwise G contains $\overline{H}_1$ or

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$\overline{H}_2$ as an induced subgraph. This shows that there can be at most six partial vertices of type 6. Every C_6-universal vertex must see any vertex of type 6 since otherwise G contains $\overline{H}_3$. Similarly any C_6-null vertex must miss all type 6 partial vertices because otherwise G contains H_1. Thus there is a homogeneous set in G consisting of the C_6 and all vertices of type 6.

Now assume that all partial vertices of C are of type 7. There cannot exist two vertices of type 7 that have the same neighbors on C because otherwise G contains $\overline{H}_3$ resp. $\overline{H}_2$ when these two vertices are non-adjacent resp. adjacent. Therefore at most three partial vertices of type 7 are possible.

If G does not contain any other vertex then $|G| \leq 9$ and we are done. Any C-universal vertex must also see either all the partial vertices or none of them because otherwise G contains $\overline{H}_2$ or $\overline{H}_3$ as an induced subgraph. Now suppose that there exists a C-null vertex n that sees some C-partial vertex p. If n sees any other C-partial vertex besides p then p must be adjacent to these other vertices since otherwise G would contain a C_5. If n has no neighbor that is non-adjacent to p then p together with all its neighbors except n forms a star-cutset that separates n from the C_6. If n has a neighbor x that is non-adjacent to p then x must be a C-null or a C-universal vertex. But then p, n, x and three appropriate vertices of the C_6 induce an H_1 resp. an $\overline{H}_3$. Therefore any C-null vertex sees none of the partial vertices.

This shows that setting A as the vertices of the C_6 and B as the set of all C-partial vertices then A and B is a homogeneous pair in G.

Finally we have to show that it is not possible that all partial vertices are of type 8. There cannot exist two partial vertices of type 8 that have the same neighbors on the C_6 because otherwise G contains $\overline{H}_1$ or $\overline{H}_2$ as an induced subgraph. This shows that there can be at most six partial vertices of type 8. Every C_6-universal vertex must see any vertex of type 8 since otherwise G contains $\overline{H}_1$. Similarly any C_6-null vertex must miss all type 8 partial vertices because otherwise G contains H_1. Thus G contains a homogeneous set consisting of the C_6 and all vertices of type 8.

This finishes the proof of Claim 2. $\diamond$

If G and $\overline{G}$ do not contain an induced cycle of length at least six then G is weakly triangulated and therefore perfect. Otherwise by Claim 1 and 2 the graph G contains a homogeneous set or a homogeneous pair or a star-cutset. Since no minimal imperfect graph contains a homogeneous set or a homogeneous pair or a star-cutset and no minimal imperfect Berge graph with at most 13 vertices can exist, this concludes the proof of the theorem. $\square$

The 3-overlap graphs of the graphs H_1-H_3 appearing in Theorem 2 all contain a triangle. Therefore Theorem 1 is an immediate consequence of Theorem 2.

Since the partner graphs of H_1-H_3 contain a triangle we get as a corollary of Theorem 2 the following result that was proved by R. Hayward and W. Lenhart [13].

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Corollary 2 *If the partner graph of a Berge graph G is triangle-free then G is perfect.* □

It is easy to see that the partner graph of a C_6 or of a domino (the graph that is obtained by identifying two C_4's in an edge) is not triangle free but their 3-overlap graph is. Thus Theorem 1 is stronger than the above corollary.

4 2-overlap graphs

Exactly the same statement that we proved in the last section for the 3-overlap graphs does also hold for the 2-overlap graphs.

Theorem 3 *If the 2-overlap graph of a Berge graph G is triangle-free then G is perfect.*

As in the case for the 3-overlap graphs it is easy to see that the 2-overlap graph of an odd hole or an odd antihole is an odd hole – with the only exception of the cycle on five vertices. Therefore we get a similar corollary as in the last section which shows that the perfectness of a C_5-free graph G is already guaranteed if its 2-overlap graph is bipartite. This again yields a class of perfect graphs which can be recognized in polynomial time.

Corollary 3 *If the 2-overlap graph of a C_5-free graph G is bipartite then G is perfect.*

Proof of Theorem 3. First we observe that neither G nor $\overline{G}$ can contain a C_6, a domino, or any of the nine graphs shown in Figure 4 since the 2-overlap graph of any such graph contains a triangle.

As observed above the 2-overlap graph of an odd hole of length at least seven or an odd antihole of length at least seven is an odd antihole. Since by assumption G does not contain a C_5 this implies that G is Berge.

If neither G nor $\overline{G}$ contains an induced cycle of length at least eight then G is weakly triangulated and therefore perfect. We therefore may assume that such a cycle exists and we choose C to be the shortest induced cycle in G or $\overline{G}$ of length at least eight. By symmetry we may assume that C is contained in G.

Claim 1: *Let x be any vertex of $G - C$ that is partial on C. Then x has exactly one or two consecutive neighbors on C.*

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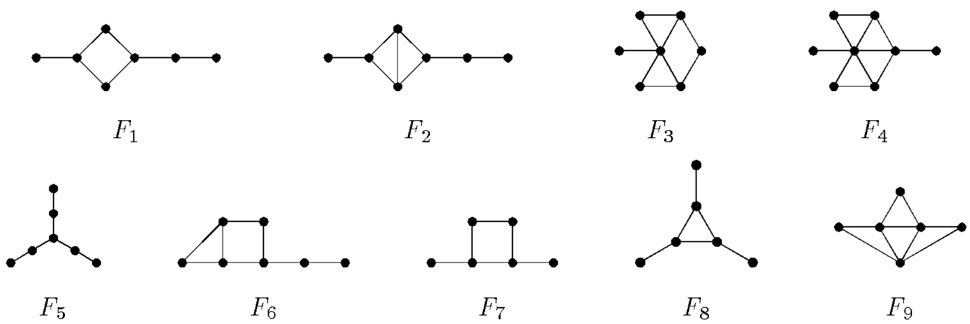

Figure 4: Graphs whose 2-overlap graph contains a triangle.

Assume that this is not true. If $N(x) \cap C$ is stable then G contains a domino or F_1 or C was not the shortest induced cycle in G of length at least eight.

Therefore x must have two consecutive neighbors on C. If x has exactly three neighbors then F_2 is contained in G or C was not the shortest induced cycle in G of length at least eight.

If x has more than three neighbors then G contains a domino or F_3 or C was not the shortest induced cycle in G of length at least eight. This finishes the proof of the claim. $\diamond$

Claim 2: *If G does not contain a homogeneous set then all vertices of $G - C$ must be C-partial.*

All C-universal vertices must see all C-partial vertices since otherwise G contains F_4 or F_9. Similarly all C-null vertices must miss all C-partial vertices since otherwise G contains F_5 or F_8. This shows that all vertices of $G - C$ must be C-partial since otherwise C together with all C-partial vertices forms a homogeneous set in G. $\diamond$

Claim 3: *If G does not contain a homogeneous set then all C-partial vertices see exactly two consecutive vertices of C or G contains a comparable pair of vertices.*

Assume that G contains neither a homogeneous set nor a comparable pair of vertices. By Claim 1 we only have to rule out the case that a partial vertex sees exactly one vertex of C.

Suppose x is a C-partial vertex that sees exactly one vertex of C. We assume that the vertices of C are labeled $a, b, c, \ldots$ and that x sees c. Since c must not dominate x there must exist a vertex y that sees x and misses c. Since G must not contain the graph F_5 as an induced subgraph, the vertex y must be adjacent to at least one of the four vertices a, b, d and e. By Claim 1 and symmetry only three cases can occur: If y sees only a then G contains a C_5. If y sees only b then G contains F_7. And finally if y sees a and b then G contains F_6. $\diamond$

Now assume that G contains no homogeneous set and no pair of comparable vertices. By Claim 3 we know that all vertices in $G - C$ must see exactly two consecutive vertices of C. Let the vertices of C be labeled $a, b, c, \ldots$ and let x be a C-partial vertex which sees the vertices c and d. Since G does not contain a comparable pair of vertices there must exist a vertex y

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that sees x and misses d. Since G must not contain the graph F_8 the vertex y must see at least one of the vertices b, c and e. By Claim 3 and symmetry there are only two cases to consider. Either y sees a and b or y sees b and c. It is easy to see that in both cases the 2-overlap graph of G contains a triangle.

Since no minimal imperfect Berge graph contains a homogeneous set or a comparable pair of vertices this finishes the proof of the theorem. □

5 1-overlap graphs

Let C be a hole of length $k \geq 7$. If k is divisible by three then the 1-overlap graph of C is the disjoint union of three $C_{k/3}$. Otherwise the 1-overlap graph of C is isomorphic to C. This implies that the 1-overlap graph of an odd cycle of length at least seven or its complement contains always an odd cycle. Therefore if G is a C_5-free graph whose 1-overlap graph is bipartite then G is Berge. The main result of this section is that these graphs are perfect.

Theorem 4 *If G contains no C_5 and the 1-overlap graph of G is bipartite then G is perfect.*

Proof. As already observed above the graph G is Berge. We will show that under the conditions of the Theorem G or $\overline{G}$ has at least one of the properties listed in Section 2 which a minimal imperfect Berge graph cannot have. This proves the perfectness of G.

Let C be the shortest even hole of length at least six in G or $\overline{G}$. Using symmetry we may assume that C is contained in G. If C does not exist then G is weakly triangulated.

We will now distinguish three different cases for the length of C:

- $\boldsymbol{|C| = 8}$

  Let the vertices of C be labeled $a, b, \ldots, h$. We will show that C is a homogeneous set. Assume not. Then there exists a C-partial vertex x. If x misses two consecutive vertices on C, then we may assume by symmetry that x sees a and misses b and c. Then $xabc$, $cdef$ and $fgha$ are P_4's that form a triangle in the 1-overlap graph, a contradiction (see Figure $5a$).

  So x does not have two consecutive non-neighbors, and we may assume by symmetry that x misses b and sees a and c. Vertex x must also see e or f, so we get a P_4 with $bcxe$ or $bcxf$. In both cases this P_4 forms in the 1-overlap graph a triangle with $defg$ and $ghab$ (see Figure $5b$), a contradiction.

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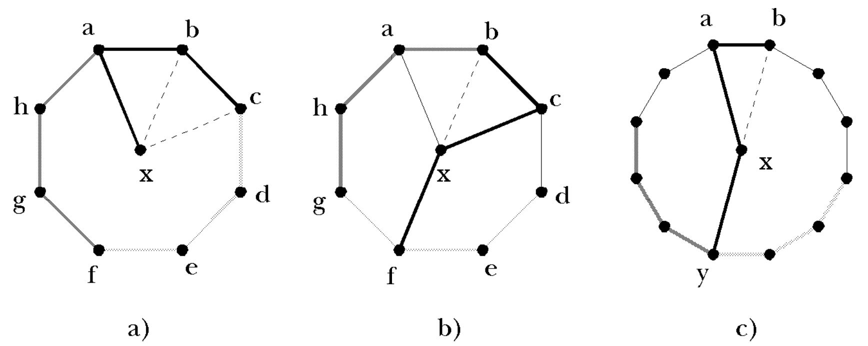

Figure 5: Triangles in the 1-overlap graphs.

- $|C| > 8$

  First we will analyse the neighborhood structure of C-partial vertices. We may assume that there is at least one C-partial vertex since otherwise C would form a homogeneous set.

  Let x be a C-partial vertex and a, b be two consecutive vertices on C such that x sees a but does not see b. Then x must miss every vertex y of C whose distances from a and from b along C are at least four, for otherwise $baxy$ plus the two P_4's along C whose endpoint is y would form a triangle in the 1-overlap graph (see Figure 5c)), a contradiction.

  Since the size of C is at least 10 there are at least two vertices in C having distance at least four from a and b along C. Thus x must have at least two consecutive non-neighbors. But this means that the neighborhood of x on C must be contained in a set of three consecutive vertices of C. Otherwise there would be an induced cycle in G of length at least six and at most $|C| - 1$, contradicting the choice of C. Thus the neighborhood of x looks like one of the four cases shown in Figure 6.

  We will now distinguish two subcases:

  α) $|C| = 6k + 4, 6k + 8,\ k \geq 1$

  Let the vertices of C be labeled $v_1, v_2, \ldots$. Let Q_1 be the P_4 on vertices $xv_1v_2v_3$, and, iteratively, given Q_i ending at vertex v_j of C let us define Q_{i+1} as $v_jv_{j+1}v_{j+2}v_{j+3}$ along C. If $|C| = 6k + 8$, then by going once around C we see that $Q_{2k+1} = v_{6k+6}v_{6k+7}v_{6k+8}v_1$, and so $Q_1, \ldots, Q_{2k+1}$ are P_4's that form an odd cycle in the 1-overlap graph. If $|C| = 6k + 4$, then by going twice around C we see that $Q_{4k+3} = v_{6k+6}v_{6k+7}v_{6k+8}v_1$ and so $Q_1, \ldots, Q_{4k+3}$ are P_4's that form an odd cycle in the 1-overlap graph.

  β) $|C| = 6k,\ k \geq 2$

  Let P, N and U denote the C-partial, C-null and C-universal vertices.

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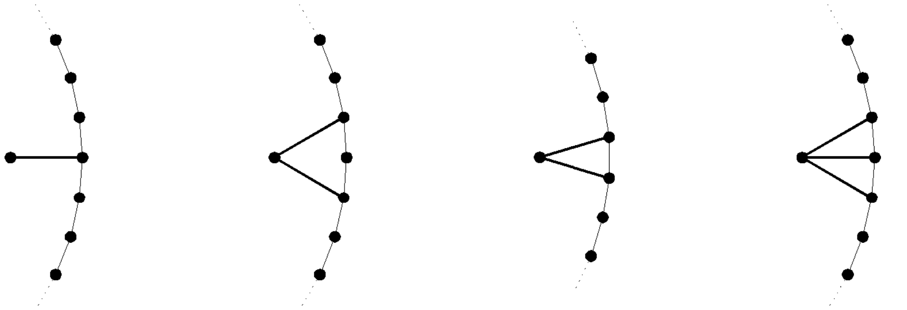

Figure 6: Possible neighborhoods of a partial vertex.

- First we show that every vertex in U must see every vertex in P. Suppose this is not the case. Let p be a partial vertex that is not adjacent to some vertex $u \in U$. Let a be a neighbor of p on C and denote by x a vertex on C that has distance at least four from a on C (note that this vertex can not be adjacent to p). Then $paux$ is a P_4 which, together with the two P_4's along C whose endpoint is a, yields a triangle in the 1-overlap graph, a contradiction.
- Next we show that the set N must be stable. Suppose this is not the case. Then take any component N_1 of N. There must be a vertex x in $P \cup U$ that is partial on two adjacent vertices n_1 and n_2 of N_1 since otherwise N_1 would be a homogeneous set. We assume that x sees n_1 and misses n_2. Let y denote any neighbor of x on C then the P_4 $n_2 n_1 x y$ and the two P_4's on C that start at y show that the 1-overlap graph of G would contain a triangle.
- Next we show that the set U cannot be empty. Assume the contrary. Let the vertices of C be labeled $a, b, c, d, e, \ldots$ and let x be any vertex of P (If P is empty then C is a homogeneous set). We may assume that x sees e and that if x has any other neighbors on C then they are f and/or g. The set $\{d, e, f, g, h\}$ must not be a cutset of G since otherwise G is not 6-connected. Therefore there must be a shortest path connecting x to the rest of the cycle C. If this path has length at least three then there exists a P_4 intersecting C in only one vertex and thus the 1-overlap graph of G would contain a triangle. Therefore the shortest path connecting x to the set $C - \{d, e, f, g, h\}$ must have length two. Let xzv be this path with z being a partial vertex and v being a vertex of C different from d, e, f, g, h. If v is none of the vertices b or c then $bcde$, $efgh$ and $vzxe$ are three P_4's intersecting in exactly one vertex (z cannot be adjacent to e since the neighborhood of every partial vertex is contained in a set of three consecutive vertices of C). If $v = b$ then $abcd$, $defg$ and $vzxe$ are three P_4's intersecting each other in exactly one vertex (again z cannot be adjacent to e). Finally if $v = c$ then if z is not adjacent to e then again $abcd$, $defg$ and $vzxe$ are three

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P_4's intersecting each other in one vertex. If z is adjacent to e then the three P_4's $xedc$, $zefg$ and $abcz$ will intersect each other in exactly one vertex. Thus in any case the 1-overlap graph of G contains a triangle.

- To finish the proof of case β) we note that if N is not empty then $N \cup C$ would form a star-cutset in $\overline{G}$ separating the two sets P and U (which are both non-empty). Thus N must be empty. But since U is completely connected to $P \cup C$ the complement of G is disconnected.

- $|C| = 6$

For the proof of this case we will make use of the following three lemmas which are proved below.

Lemma 1 *Let G be a Berge graph whose 1-overlap graph is bipartite. If G contains a C_6 such that no vertex of G is universal on this C_6 then G contains a homogeneous set or a comparable pair of vertices or a vertex of degree at most three.*

Lemma 2 *Let G be a Berge graph whose 1-overlap graph is bipartite. If G contains a C_6 and a C_6-universal vertex u that is partial on the C_6-partial vertices, then G contains a comparable pair of vertices.*

Lemma 3 *Let G be a Berge graph whose 1-overlap graph is bipartite. If G contains a C_6 such that at least two vertices of G are not C_6-partial then G contains a homogeneous set or a homogeneous pair or a comparable pair of vertices.*

Now we can finish the proof of the case $|C| = 6$ by just combining these three lemmas appropriately. Let C denote the C_6 and denote by P, U and N the C-partial, C-universal and C-null vertices.

If U is empty then the proof follows immediately from Lemma 1. If the union of U and N contains at least two elements then we are done because of Lemma 3.

Thus the only case we have to check is when N is empty and U contains exactly one vertex u. By Lemma 2 vertex u must either be adjacent to all vertices of G – which implies that the complement of G is disconnected – or has exactly the vertices of C as its neighbors. But then the vertex u would not be contained in a clique of size four. □

Before proving the three lemmas we need to introduce the notion of a *good set*. Let $C = abcdef$ be an induced C_6 in some graph G. Let H be a set consisting of the vertex b and of (some) C-partial vertices in G that are adjacent to a and c but do not see d, e or f. If H contains at least three vertices and is not a clique, then we call H a good set for C. A graph is said to contain a good set, if there exists some induced C_6 for which a good set exists.

The following result will be an important tool in the proofs of Lemma 1 and Lemma 3.

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The good set argument *If the 1-overlap graph of a Berge graph G is triangle-free and G contains a good set then G contains a homogeneous set.*

Proof. Let $C = abcdef$ be a C_6 in G and let H be a maximal good set with respect to C such that every vertex of H sees a and c but misses d, e, and f.

Denote by H' the vertices of a component of the complement of H with $|H'| \geq 2$. Since H contains at least two vertices and is not a clique such a component must exist. If the vertices of H' form a homogeneous set in G then we are done. Otherwise there must exist a vertex i, not in H', that is partial on two non-adjacent vertices of H'. Let x and y denote these non-adjacent vertices and assume that i sees x but does not see y. Note that by the definition of H', vertex i cannot be a vertex of H. Let z be some vertex of H different from x and y. Now i must see c since otherwise there are the three P_4's $ixcy$, $zcde$ and $efay$. By symmetry i must also see a. Now i sees the vertices a and c and since i is not a vertex of H it must also see at least one of the vertices d, e and f.

If i sees the vertex e then the three P_4's $eicy$, $dcxa$ and $efaz$ would form a triangle in the 1-overlap graph of G. Therefore i cannot be adjacent to e but must see d or f. By symmetry we may assume that i sees d. But then the three P_4's $edix$, $efaz$, and $aycd$ show that the 1-overlap graph of G would contain a triangle. □

Proof of Lemma 1 Let $C = abcdef$ denote the C_6 such that no vertex of G is universal on this C_6. We will show that the assumption that G contains neither a homogeneous set nor a comparable pair of vertices nor a vertex of degree at most three leads to a contradiction. Recall from figure 2 the possible types of neighborhood along a C_6 for a C_5-free graph. We are now going to eliminate successively all these types.

Claim 1: *No C-partial vertex is of type* 8.

Let x denote a partial vertex that is of type 8, i.e. that sees exactly the vertices a, b, c, d and e in C. Then there must exist a vertex y that is adjacent to f and non-adjacent to x since otherwise f is dominated by x. Now the five P_4's $abcd, xafy, cdef, bafy$ and $fexc$ show that y must be adjacent to at least one of the vertices a or b.

Let us first assume that y is adjacent to a. Then the five P_4's $cxay, fexb, abcd, exay$ and $bcde$ show that cy or ey must be an edge. If cy is an edge then the vertices $yfexc$ induce a C_5 in G which implies that y must be adjacent to e. This shows that in any case y must be adjacent to e. But then the five P_4's $abcd, fexb, cxay, bafe$ and $yexc$ show that y must also be adjacent to c. Now the five P_4's $fycx, eyab, cxaf, eycb$ and $defa$ imply that y must see the vertex b. By symmetry y must also see d but now y is C-universal which contradicts the assumption of the lemma.

Now assume that y is not adjacent to a and, by symmetry, also not adjacent to e. As shown above y must then be adjacent to b. But now the vertices $ybxef$ induce a C_5 in G. ◇

Claim 2: *No C-partial vertex is of type* 6.

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Let x denote a partial vertex that is of type 6, i.e. that sees exactly the vertices a, c, d and e in C. Then there must exist a vertex y that is adjacent to f and non-adjacent to x since otherwise f is dominated by x.

Now looking at the five P_4's $xafy$, $abcd$, $fexc$, $dxab$, and $cdef$ shows that ya must be an edge in G. Then the five P_4's $exay$, $abcd$, $fexc$, $dxab$, and $cdef$ show that ye must be an edge. This new edge now gives rise to the five P_4's $ayed$, $excb$, $dxaf$, $bcde$, and $cxaf$ which show that y must be adjacent to d.

This shows that y has four consecutive neighbors on C. Therefore y must either be of type 8 or it must be C-universal. By Claim 1 and the assumption of the lemma both cases are impossible. $\diamond$

Claim 3: *No C-partial vertex is of type 7.*

Let x denote a partial vertex that is of type 7, i.e. that sees exactly the vertices a, b, d and e in C. Then there must exist vertices y and z such that y sees c and not x and z sees f and not x. If such vertices would not exist then x dominates c or f.

First we analyze the case that y equals z. Then $abcyf$ would form a C_5 in G which implies (using symmetry) that ya must be an edge. Now $dxayc$ induces a C_5 in G which shows that y must be adjacent to d. By Claim 1 and the assumption of the lemma the vertex y cannot have any other neighbor on C. But now the five P_4's $abcd, fydx, fabc, bxdy$ and $efyc$ show that the 1-overlap graph of G contains a C_5.

Now assume that y and z are different vertices. The three P_4's $xbcy, xafz$ and $cdef$ show that either yb or za must be an edge in G. By symmetry we may assume that yb is an edge. Then the five P_4's $abcd, bxef, ycdx, fabc$ and $exby$ show that y must be adjacent to d or e. If y is adjacent to d then the five P_4's $axdy$, $fabc$, $edyb$, $faxd$, and $bcde$ show that y must have at least one more neighbor on C. But this cannot be possible because otherwise y is universal or of type 8 or 6 which contradicts Claim 1 resp. Claim 2.

Thus we know that y cannot be adjacent to d and therefore must see e. Now since y must be a partial vertex, the only possible case now is that y is of type 7 and therefore must see f, but is non-adjacent to a and d. But now the five P_4's $bcde$, $cyfa$, $byed$, $fabc$, and $axey$ show that the 1-overlap graph of G contains an induced C_5. $\diamond$

Claim 4: *No C-partial vertex sees two opposite vertices (i.e., vertices at distance three) of C.*

This is a simple observation, since the only partial vertices seeing two opposite vertices of C are of type 6, 7 or 8. But these cases are excluded by Claims 1-3. $\diamond$

Claim 5: *No C-partial vertex is of type 5.*

Let x denote a partial vertex that is of type 5, i.e. that sees exactly the vertices a, c and e in C. Let y be a vertex adjacent to a and different from x, b and f. The vertex y must exist since otherwise a has degree three. By Claim 4 y cannot be adjacent to d.

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Let us first assume that y does not see x. Then the five P_4's $yabc$, $cdef$, $exay$, $fabc$, and $axed$ show that y must have at least one more neighbor on C.

If y is adjacent to b then the five P_4's $ybcx$, $cdef$, $exay$, $fabc$ and $axed$ show that y must be adjacent to c (by Claim 4 y cannot be adjacent to e). But now the five P_4's $ycde$, $cxaf$, $bcde$, $exay$, and $fabc$ show that the 1-overlap graph of G would contain a C_5.

Therefore y cannot be adjacent to b and by symmetry it is also non-adjacent to f. Thus we can assume that y is adjacent to c. Now the five P_4's $fayc$, $bcde$, $exay$, $fabc$, and $axed$ show that y must be adjacent to e. But then again we have five P_4's, namely $fabc$, $axed$, $cyaf$, $excb$, and $ayed$ which show that the 1-overlap graph of G contains a C_5.

Now we are studying the case when y is adjacent to x. The five P_4's $yxef$, $axcd$, $yafe$, $yxcd$ and $fabc$ show that y must have at least one other neighbor on C. By Claim 4 we know that y cannot be adjacent to d and using symmetry we may assume that y is adjacent to c or b.

If y is adjacent to c then by Claim 4 it is not adjacent to f. Now the five P_4's $ycde$, $fabc$, $yxef$, $aycd$ and $exab$ imply that y must see e. But then $fabc$, $axed$, $eycb$, $cxaf$, and $ayed$ show that the 1-overlap graph of G contains a C_5.

Therefore we know that y is not adjacent to c and by symmetry also not adjacent to e. Then we may assume that yb is an edge. But now the five P_4's $yxcd$, $exab$, $cdef$, $byxe$, and $fabc$ show again that the 1-overlap graph of G contains a C_5. ◇

Claim 6: *No C-partial vertex is of type 4.*

Let x denote a partial vertex that is of type 4, i.e. that is adjacent to a, b and c. Then there must exist vertices y and z with y being adjacent to b but non-adjacent to x and z adjacent to x and non-adjacent to b since otherwise either b dominates x or x dominates b.

Now the three P_4's $zxby$, $axcd$ and $efab$ show that yz must be an edge. Then we find the three P_4's $cbyz$, $dcxa$ and $efab$ which imply that c must be adjacent to y or z. By symmetry we may assume that c is adjacent to y. Now the five P_4's $faby$, $dcxa$, $efab$, $ycxa$ and $cdef$ show that y must see a (note that by Claim 4 y cannot see f). Since y cannot have any other neighbor on C we can apply the good set argument to the set $\{b, x, y\}$ which shows that G contains a homogeneous set. ◇

Claim 7: *No C-partial vertex is of type 3.*

By the preceding claims we know that any C-partial vertex has at most two neighbors in C. Let x denote a partial vertex that is of type 3, i.e. that is adjacent to a and c. Then there must exist a vertex y that is adjacent to b and not adjacent to x since otherwise x dominates b. Now the five P_4's $ybcx$, $bafe$, $axcd$, $faby$ and $cdef$ show that (using symmetry) y must be adjacent to at least one of a and f.

If yf is an edge then we find the five P_4's $abcd$, $efyb$, $cxaf$, $bcde$ and $xafe$ which imply that ye must be an edge. But this contradicts Claim 4.

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Therefore we know that ya must be an edge. Now the five P_4's $bcde$, $xafe$, $ybcx$, $cdef$ and $cxay$ show that yc must be an edge. But this means that y has more than two neighbors on C contradicting the preceding claims. ◇

Note, that so far we have shown by Claims 1-7 that any C-partial vertex may see only one or two consecutive vertices of C. We will make use of this fact in the proof of the following claim.

Claim 8: *No partial vertex is of type* 2.

Let x denote a partial vertex that is of type 2, i.e. that is adjacent to a and b. Let y be any vertex adjacent to c different from b and d and let z be any vertex adjacent to f different from a and e. By Claim 4 the vertices y and z must be different. Now the three P_4's $cdef$, $abcy$ and $xafz$ show that at least one of the three edges yb, za or zx must exist.

Let us first assume that zx is an edge. Then z must also be adjacent to e since otherwise G would contain an induced C_7. But now we have the five P_4's $cdef$, $bafz$, $xzed$, $bafe$ and $zxbc$ — a contradiction.

Let us now assume that yb or za is an edge. By symmetry we may assume that za is an edge in G. Now we find the three P_4's $ycde$, $xafe$ and $zabc$ which imply that y must see d. Let w denote any neighbor of e different from d and f. The vertex w must exist since otherwise e has degree three. Then the three P_4's $cdew$, $xbcy$ and $xafe$ show that w must be adjacent to d. Now let u be a vertex that sees x but does not see a. Such a vertex must exist since otherwise a would dominate x. Now we have the three P_4's $zabc$, $uxaf$ and $cdef$ which show that u must be adjacent to f. But then the three P_4's $cdef$, $fuxb$ and $zabc$ force u also to be adjacent to b which contradicts the above stated claims. ◇

So far we have shown by Claims 1-8 that every C-partial vertex has only one neighbor in C. Since every vertex of C must have a partial vertex as its neighbor we can choose vertices x, y and z with x being adjacent to a, y being adjacent to c and z being adjacent to e. Then we have the three P_4's $xabc$, $ycde$ and $zefa$ which show that the 1-overlap graph of G contains a triangle. □

Proof of Lemma 2 Let $abcdef$ be the vertices of the C_6. We will show that if G contains no comparable pair of vertices then no C_6-universal vertex can be partial on the C_6-partial vertices.

Claim 1: *Every universal vertex is adjacent to all partial vertices of type 2, 4, 5, 6, and 7.*

Let u be a C_6-universal vertex and p be a C_6-partial vertex not adjacent to u. We will show that in all these five cases for the vertex p there are five P_4's in G such that the 1-overlap graph of G contains a C_5.

Let p be of type 2, i.e., p sees the vertices a and b. Then take the five P_4's $abcd$, $eubp$, $fabc$, $dubp$ and $pafe$.

Let p be of type 4, i.e., p sees the vertices a,b and c. Then take the five P_4's $eubp$, $cpaf$, $bcde$, $fubp$ and $apcd$.

Let p be of type 5, i.e., p sees the vertices a,c and e. Then take the five P_4's $cdef$, $duap$, $bafe$, $apcd$ and $peub$.

Let p be of type 6, i.e., p sees the vertices a,b,c and e. Then take the five P_4's $bafe$, $fucp$, $aped$, $fubp$ and $apcd$.

Let p be of type 7, i.e., p sees the vertices a,b,d and e. Then take the five P_4's $peuc$, $bafe$, $apdc$, $bpef$ and $abcd$. ◇

Claim 2: *Any universal vertex is adjacent to all partial vertices of type 3.*

Let u be a universal vertex and p be a partial vertex of type 3 not adjacent to u. We assume that p sees the vertices a and c.

Since the vertex b must not dominate p there must exist a vertex x that sees p but does not see b. Then the three P_4's $xpcb$, $cdef$ and $euap$ show that xc must be an edge. By symmetry also xa must be an edge. Now the three P_4's $axcd$, $bafe$, and $fucp$ show that xd must be an edge. But this gives a contradiction since now we have the three P_4's $dxab$, $euap$, and $cdef$. ◇

Claim 3: *A universal vertex that is non-adjacent to a partial vertex of type 1 cannot see any partial vertex.*

Let u be a universal vertex and p be a partial vertex of type 1 not adjacent to u where we assume that p sees the vertex a. Let x be a C_6-partial vertex that sees u. We will show that this gives a contradiction.

First assume that x is adjacent to b and f. Then the three P_4's $pauc$, $defa$ and $fxbc$ show that x must be adjacent to c. By symmetry it must also be adjacent to e. But now the three P_4's $cxfa$, $paue$ and $bcde$ and the three P_4's $fabc$, $bxed$ and $paue$ show that x must also be adjacent to a and d. This contradicts the assumption that x is a C_6-partial vertex.

Now let us assume that x sees b but does not see f. Then the three P_4's $fabx$, $pauc$, and $cdef$ show that xa must be an edge. Then we can conclude that xe must be an edge because of the P_4's $xafe$, $paud$, and $bcde$. But then the three P_4's $paue$, $abcd$ and $bxef$ imply that x must see f — a contradiction.

Thus we know that x cannot see b and by symmetry can also not see f. If x sees a then the three P_4's $xafe$, $paud$ and $bcde$ imply that x must see e. But this gives a contradiction since then the P_4's $exab$, $pauc$ and $cdef$ imply that x must see b.

Therefore we know that x sees neither a nor b nor f. If x sees e then the three P_4's $bcde$, $xefa$ and $paud$ give a contradiction.

By symmetry x can also not see c and therefore d is the only possible neighbor of x. But if x sees d then we find the three P_4's $bcdx$, $paud$ and $bafe$ which again gives a contradiction. ◇

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Claim 4: *A universal vertex that is non-adjacent to a partial vertex of type 8 cannot see any partial vertex.*

Let u be a universal vertex and p be a partial vertex of type 8 which sees the vertices a, b, c, d and e and does not see u. Assume that there exists a partial vertex x that is adjacent to u.

Let us first assume that x sees a and e. Then the P_4's $axed$, $fabc$ and $fudp$ show that x must see d and by symmetry it also must see b. Now the three P_4's $dxaf$, $fubp$ and $bcde$ resp. $axdc$, $bafe$ and $fucp$ show that x must also see f and c. But this is a contradiction since x is assumed to be a C_6-partial vertex.

Now assume that x sees a but does not see e. Then the three P_4's $xafe$, $fudp$ and $abcd$ show that xf must be an edge. Then x must also see d because of the P_4's $defx$, $abcd$ and $fubp$. But this gives a contradiction because we now have the three P_4's $axde$, $fabc$ and $fudp$.

Thus we know that x cannot see a and by symmetry also not e. Let us assume that xf is an edge. Then the P_4's $bafx$, $fucp$ and $bcde$ imply that x must see b and by symmetry it also must see d. But then the three P_4's $fubp$, $cdef$ and $dxba$ give a contradiction.

Thus we know that x sees neither a nor e nor f. Then x cannot see b because of the three P_4's $bcde$, $fabx$, and $fudp$. By symmetry x can also not see d.

Thus the only possible neighbor of x is c. But if x sees c then there are the three P_4's $fucp$, $bafe$ and $xcde$ giving the desired contradiction. ◇

We are now going to complete the proof of Lemma 2. If a C_6-universal vertex u misses a C-partial vertex of type 1 or 8, then by Claims 3 and 4 vertex u must miss all of P. On the other hand if u sees all partial vertices of type 1 and 8 then by Claims 1 and 2 it sees all of P.

□

Proof of Lemma 3 Assume that G contains neither a homogeneous pair nor a homogeneous set nor a comparable pair of vertices. We will show that this assumption leads to a contradiction.

Let $C = abcdef$ denote the C_6 and let P denote the set of C-partial vertices. Then there must exist a C-universal or C-null vertex that is P-partial, otherwise the sets C and P would form a homogeneous pair. By Lemma 2 we know that no C-universal vertex can be P-partial. Thus there must exist a C-null vertex that is P-partial. We will show that this is not possible yielding the desired contradiction.

Claim 1: *No C-null vertex can see a C-partial vertex of type* $2, 3, 6$ *or* 7.

Let n be a C-null vertex and p be a C-partial vertex adjacent to n. We will show that if p is of type $2, 3, 6$ or 7 then the 1-overlap graph of G contains a C_5.

If p is of type 2, i.e., p sees a and b then there are the five P_4's $abcd$, $npaf$, $cdef$, $npbc$ and $pafe$.

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If p is of type 3, i.e., p sees a and c then there are the five P_4's $pafe$, $npcb$, $cdef$, $npaf$ and $abcd$.

If p is of type 6, i.e., p sees a, b, c and e then there are the five P_4's $cpaf$, $nped$, $abcd$, $npaf$ and $bped$.

If p is of type 7, i.e., p sees a, b, d and e then there are the five P_4's $fabc$, $npdc$, $bpef$, $apdc$ and $npef$. $\diamond$

Claim 2: *No C-null vertex can see a C-partial vertex of type 5.*

Let n be a C-null vertex and p be a C-partial vertex that is adjacent to n and of type 5, i.e., p sees a, c and e.

Since G does not contain a comparable pair of vertices there must exist a vertex x that sees b but does not see p. Then the three P_4's $pabx$, $npef$ and $bcde$ imply that xa must be an edge. By symmetry also xc must be an edge. Now we find the three P_4's $cxaf$, $npab$ and $bcde$ which imply that x must see f. Now we get a contradiction since there are the three P_4's $fxcp$, $nped$ and $abcd$. $\diamond$

Claim 3: *No C-null vertex can see a C-partial vertex of type 4.*

Let n be a C-null vertex and p be a C-partial vertex that is adjacent to n and of type 4, i.e., p sees a, b and c.

Since G does not contain a comparable pair of vertices there must exist a vertex x that sees b but does not see p. Then the three P_4's $npbx$, $cpaf$ and $bcde$ imply that x must see n. Now we find the three P_4's $abxn$, $dcpn$ and $defa$ which imply that xa must be an edge. By symmetry also xc must be an edge.

Now the good set argument shows that x must see at least one of the vertices d, e or f. But x cannot see d because otherwise there are the three P_4's $pbxd$, $npaf$ and $cdef$. By symmetry x can also not see f.

Thus x must see e but then we find the three P_4's $pnxe$, $cpaf$ and $bcde$. $\diamond$

Claim 4: *If a C-null vertex is adjacent to a partial vertex of type 1 then it must be adjacent to all partial vertices.*

Let n be a C-null vertex and p be a C-partial vertex that is adjacent to n and of type 1, i.e., p sees a. Let x denote a C-partial vertex that is not adjacent to n.

First let us assume that x sees the vertices c and e. Then there are the three P_4's $cxef$, $fapn$ and $abcd$ which imply that x must be adjacent to f. By symmetry x must also be adjacent to b. Next we find the three P_4's $exba$, $fapn$ and $cdef$ which imply that x must see a. Then there are the three P_4's $npax$, $fabc$ and $bxed$ which imply that x must be adjacent to p (note that x is a partial vertex and therefore cannot be adjacent to d). But now we get a contradiction with the three P_4's $npaf$, $pxed$ and $abcd$.

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Now we assume that x sees neither c nor e. If x is adjacent to d then there are the three P_4's $xdef$, $abcd$ and $fapn$ which show that x must see f. By symmetry x must also see b. But then the three P_4's $defa$, $bapn$ and $fxbc$ give a contradiction. Thus x cannot be adjacent to d. Moreover x also cannot be adjacent to f because otherwise there are the three P_4's $defx$, $abcd$ and $fapn$. By symmetry x can also not be adjacent to b. Thus x must be adjacent to a but then there are the three P_4's $bcde$, $bapn$ and $xafe$.

Thus we know that x must be adjacent to exactly one of the two vertices c and e. By symmetry we may assume that x sees c and misses e. Vertex x must miss f or else $efxc$, $fapn$ and $abcd$ are three P_4's inducing a triangle in the 1-overlap graph. Then x must miss d or else $xdef$, $fapn$ and $abcd$ are P_4's that form a triangle in the 1-overlap graph. Then x must miss a or else $xafe$, $bapn$ and $bcde$ are P_4's that form a triangle. Then x must see b or else $abcx$, $fapn$ and $cdef$ are P_4's that form a triangle. Now there must exist a vertex y that is adjacent to x and non-adjacent to b since otherwise G contains a comparable pair of vertices. The vertex y must be different from p because otherwise G would contain the C_7 $xcdefay$ Then there are the three P_4's $pafe$, $xcde$ and $yxba$ which show that y must see a. Now the three P_4's $xcde$, $pabc$ and $xyaf$ imply that y is adjacent to f. But then the three P_4's $bxyf$, $defa$ and $bapn$ give a contradiction. $\diamond$

Claim 5: *If a C-null vertex is adjacent to a partial vertex of type 8 then it must be adjacent to all partial vertices.*

Let n be a C-null vertex and p be a C-partial vertex that is adjacent to n and of type 8, i.e., p sees a, b, c, d and e. Let x denote a C-partial vertex that is not adjacent to n.

First let us assume that x sees the vertices a and e. Then the three P_4's $exab$, $cdef$ and $fapn$ show that x must be adjacent to b. By symmetry x must also be adjacent to d. Moreover the three P_4's $fapc$, $bcde$ and $npex$ show that x must also see p. Now the three P_4's $dxaf$ $npef$ and $bcde$ show that x must be adjacent to f. But then we get a contradiction since the three P_4's $exbc$, $fxpn$ and $defa$ imply that x must see c which is not possible since x is assumed to be a C-partial vertex.

Next let us assume that x sees neither a nor e. If x sees f then there are the three P_4's $defx$, $abcd$ and $fapn$ which show that x must also see d. By symmetry x must also see b. But now the three P_4's $npef$, $afxd$ and $bcde$ give a contradiction. Thus x cannot see f. If x sees b then the three P_4's $fabx$, $bcde$ and $npef$ give a contradiction. By symmetry x can also not see d. Thus x must see c but then there are the three P_4's $abcx$, $cdef$ and $npaf$.

Therefore we know that x is adjacent to exactly one of the two vertices a and e. By symmetry we may assume that x sees a. Then the three P_4's $xapn$, $abcd$ and $bpef$ imply that xp must be an edge. Now xd cannot be an edge because otherwise we find the three P_4's $npef$, $axde$ and $fabc$. Similarly x cannot be adjacent to c or f because of the P_4's $xcde$, $fabc$ and $npef$ resp. $defx$, $fapn$ and $abcd$. Hence three P_4's $xabc$, $cdef$ and $fapn$ imply that x must see b.

Since G does not contain a comparable pair of vertices there must exist a vertex y that is adjacent to c but is non-adjacent to p. Then the three P_4's $epcy$, $fapn$ and $dcba$ imply that y must see e. Now the three P_4's $cyef$, $dcbx$ and $fapd$ imply that y is adjacent to f. But then

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the three P_4's $fycp$, $edcb$ and $xafe$ yield a contradiction. $\diamond$

$\square$

6 Comparison with known classes of perfect graphs

In this section we analyze the relation of the new classes of perfect graphs that we have introduced in this paper, with the known classes of perfect graphs.

For brevity of notation we will denote our new classes defined via 3-, 2-, and 1-overlap graphs as $\mathcal{O}_3$,$\mathcal{O}_2$ and $\mathcal{O}_1$, i.e. $\mathcal{O}_3$ denotes the class of all Berge graphs whose 3-overlap graph is triangle-free; $\mathcal{O}_2$ denotes the class of all Berge graphs whose 2-overlap graph is triangle-free and $\mathcal{O}_1$ denotes the class of all C_5-free graphs whose 1-overlap graph is bipartite.

First we note, that all three of these classes contain the C_8 and its complement. Therefore the classes $\mathcal{O}_3$, $\mathcal{O}_2$ and $\mathcal{O}_1$ are neither contained in the class of *strict quasi parity* graphs [19] nor in the class BIP* [4] nor *strongly perfect* [2]. Two vertices are called an *even pair* if all induced paths connecting these two vertices have even length. A graph G is called *quasi parity* if every induced subgraph or its complement contains an even pair. Meyniel [19] proved that quasi parity graphs are perfect. The graph of Figure 7a) shows that the class $\mathcal{O}_3$ is not quasi parity.

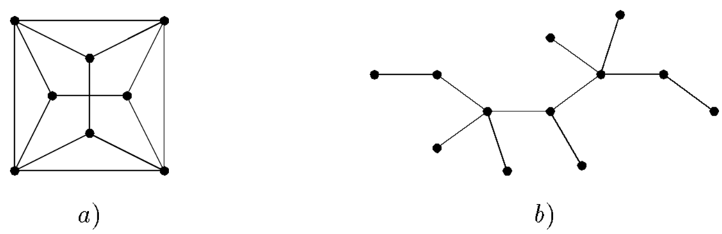

Figure 7:

For the graphs in $\mathcal{O}_1$ we do not know whether these graphs are quasi parity graphs. However, the following Lemma shows that the class $\mathcal{O}_2$ is contained in the quasi parity graphs.

Lemma 4 *If G is a Berge graph whose 2-overlap graph is triangle-free then G is quasi parity.*

Proof. It is easy to see that if G is a Berge graph that contains a homogeneous set, or a comparable pair of vertices, or a vertex of degree two, then G or $\overline{G}$ contains an even pair. Moreover as shown in [12], every weakly triangulated graph contains an even pair.

Suppose the statement of the lemma is not true, i.e., there exists a Berge graph G whose 2-overlap graph is triangle-free, but G is not quasi parity. Then G cannot be weakly triangulated and Claims 1, 2 and 3 of the proof of Theorem 3 show that G must consist of a cycle C of

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length at least eight where all vertices of $G - C$ have exactly two consecutive vertices of C as its neighbors.

Let the vertices of C be labeled $a, b, c, \ldots$ and let x be a vertex of $G - C$ that sees b and c. Let y be a neighbor of d. If the vertex y does not exist then the vertex d has degree two and therefore as noted above G contains an even pair. If the second neighbor of y on C is e then independently whether x and y are adjacent, the graph induced by the vertices a, b, c, d, e, f, x, y contains a triangle in the 2-overlap graph. Therefore the second neighbor of y on C must be c. Using the same argument once more with y in the role of x, we see that there must exist a vertex z adjacent to e and d. But as we have seen just above, such a vertex cannot exist in G. $\square$

Finally we want to analyze which classes of perfect graphs are contained in our classes $\mathcal{O}_3$, $\mathcal{O}_2$ and $\mathcal{O}_1$. A graph is called *P_4-reducible*, if every vertex is contained in at most one P_4 [16]. Trivially all the three classes contain the class of P_4-reducible graphs. A graph is called *P_4-sparse*, if any set of five vertices induces at most one P_4. Clearly, every P_4-reducible graph is P_4-sparse. It is easy to see that the class $\mathcal{O}_3$ contains the P_4-sparse graphs and a simple case analysis shows that the same is true for the class $\mathcal{O}_1$. However the class $\mathcal{O}_2$ does not contain the P_4-sparse graphs; the graph F_8 of Figure 4 is a counterexample.

The graph of Figure 7b) shows that none of the three classes $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ contains the trees, the interval graphs or the permutation graphs (see [10] for definitions). However, as noted before Corollary 2, the class $\mathcal{O}_3$ contains all graphs whose partner graph is triangle-free.

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