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Perfectness is an Elusive Graph Property

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Abstract

A graph property is called elusive (or evasive) if every algorithm for testing this property has to read in the worst case $\binom{n}{2}$ entries of the adjacency matrix of the given graph. Several graph properties have been shown to be elusive, e.g. planarity [3] or k-colorability [5]. A famous conjecture of Karp says that every non-trivial monotone graph property is elusive. We prove that a non-monotone but hereditary graph property is elusive: perfectness.

1 Introduction

Given a graph property, consider the following two-players game to define elusiveness. Player **A** wants to know whether an unknown simple graph on a given node set has the graph property in question by asking Player **B** one by one, whether a certain pair of nodes is an edge. At each stage Player **A** makes full use of the information of edges and non-edges he has up to that point in order to decide whether the graph has the property. Player **A** wants to minimize the number of questions, Player **B** wants to force him to ask as many questions as possible. The number of questions needed for the decision if both players play optimally from their point of view is the complexity of the studied graph property. If there is a strategy enabling Player **B** to force Player **A** to test *every* pair before coming to a decision, the property is said to be **elusive** (or also evasive).

Several graph properties are known to be elusive, e.g., having a clique of a certain size or a coloring with a certain number of color classes (BOLLOBAS [4])

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and being planar for graphs on ≥ 5 nodes (Best et al. [3]), see [5, 11] for more examples. Aanderaa and Rosenberg conjectured [3] that there exists some $\gamma > 0$ such that the complexity of every non-trivial monotone graph property (i.e., a property preserved under deleting edges) is at least γn^2. This conjecture has been proved by Rivest and Vuillemin for $\gamma = \frac{1}{16}$. The value of γ has been improved over the years. Currently the largest value of γ for which the conjecture of Aanderaa and Rosenberg is known to be true is $\frac{1}{4} - o(1)$. This result was established by Kahn, Saks and Sturtevant [8]. A sharpened version of the Aanderaa-Rosenberg conjecture is due to Karp. He conjectures that every non-trivial monotone graph property is elusive.

The subject of the present paper is to prove that perfectness is elusive: a property which is not monotone but hereditary (preserved under deleting nodes) and concerned to the relation of maximum cliques and optimal colorings. Perfect graphs behave nicely from an algorithmic point of view [6] and have interesting relationships to surprisingly many other fields of scientific enquiry [1]. However, the recognition problem for perfect graphs is unsolved and its complexity is even unknown. The present paper contributes some information by showing: *perfectness is a graph property as complex as possible, namely, elusive.*

Berge [2] proposed to call a graph G **perfect** if, for each of its (node-induced) subgraphs $G' \subseteq G$, the chromatic number equals the clique number (i.e., if we need as many stable sets to cover all nodes of G' as a maximum clique of G' has nodes). This means that identifying one induced imperfect subgraph would enable Player **A** to make the final decision: the graph in question is not perfect. For that, so-called **minimally imperfect graphs** are of particular interest (that are imperfect graphs all of their proper subgraphs are perfect). The only known examples of minimally imperfect graphs are chordless odd cycles of length ≥ 5, termed **odd holes**, and their complements, called **odd antiholes**. (A famous conjecture due to Berge [2] says that odd holes and odd antiholes are the only minimally imperfect graphs.) Consequently, Player **B** has to answer in such a way that no minimally imperfect subgraph appears until Player **A** asks the last question but that the last answer can create a minimally imperfect subgraph.

The idea for providing a strategy to Player **B** goes as follows. Find perfect graphs such that you cannot reach another perfect graph by deleting or adding one edge. We call an edge e of a perfect graph G **critical** if $G - e$ is imperfect. Analogously, we call an edge e not contained in a perfect graph G **anticritical** if $G + e$ is imperfect. A perfect graph G without isolated nodes is **critically perfect** if G has only critical edges. The complement of a critically perfect graph is again perfect (due to Lovász [9]) and has

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the property that adding an edge not contained in the graph so far yields an imperfect graph. We call the complements of critically perfect graphs **anticritically perfect**. We look for **bicritically perfect graphs** which are both critically *and* anticritically perfect: the deletion and addition of an *arbitrary* edge yields an imperfect graph. Hence, our task is:

Problem 1 *Find, for as many n as possible, a bicritically perfect graph G_n on n nodes.*

If there exists a bicritically perfect graph G_n, then Player **B** has only to answer all but the last question "$ij \in E$?" of Player **A** as in G_n. I.e., Player **B** has only to apply the following strategy for graphs on n nodes.

Strategy 1 Let G_n be a bicritically perfect graph on n nodes.
For questions 1 to $\binom{n}{2} - 1$:
Answer "$ij \in E$?" with YES if $ij \in E(G_n)$, NO otherwise.

Then no induced imperfect subgraph appears during the first $\binom{n}{2} - 1$ questions, and the answer to the last question yields the decision:

$$\text{Answer "}ij \in E\text{?" with } \begin{cases} \text{YES} & \text{if } ij \in E(G_n) & \text{then the graph is perfect} \\ \text{NO} & \text{if } ij \in E(G_n) & \text{then the graph is imperfect} \\ \text{YES} & \text{if } ij \notin E(G_n) & \text{then the graph is imperfect} \\ \text{NO} & \text{if } ij \notin E(G_n) & \text{then the graph is perfect} \end{cases}$$

A first step towards Problem 1 was a computer search of HOUGARDY [7] enumerating which perfect graphs on upto 10 nodes are critically perfect.

Theorem 2 (HOUGARDY [7]) *No critically perfect graphs with fewer than 9 nodes exist. On 9 and 10 nodes there are precisely 3 and 10 critically perfect graphs, respectively.*

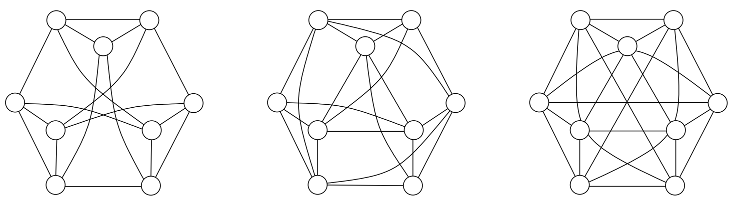

Figure 1: The three smallest critically perfect graphs.

This paper appeared in: SIAM Journal on Computing 34:1 (2004), 109-117

Clearly, Theorem 2 remains true if "critically perfect" is replaced by "anticritically perfect". Figure 1 shows the three critically perfect graphs on nine nodes. The first graph is self-complementary and, therefore, also anticritical. I.e., it is our first example G_9 of a bicritically perfect graph. The other two graphs are not anticritical, but only their complements are. Every of the critically perfect graphs with ten nodes is not anticritical (see next section). That means particularly: there are no bicritically perfect graphs G_n with $n \leq 8$ and $n = 10$. In Section 2, we present a technique of constructing examples of bicritically perfect graphs that bases on the characterization of critically and anticritically perfect line graphs. In Section 3, we apply the knowledge from the previous section to construct the studied graphs G_n if $n \geq 12$. Section 4 provides a slightly different strategy for the cases $n = 10, 11$. We conclude with some remarks on the cases $n \leq 8$.

2 Bicritically Perfect Line Graphs

This section provides characterizations of critically or anticritically perfect line graphs established in [12, 13]. We obtain the **line graph** $L(F)$ of a graph F by taking the edges of F as nodes of $L(F)$ and joining two nodes of $L(F)$ by an edge iff the corresponding edges of F are incident. It is well-known that the line graph $L(F)$ of a graph F is perfect iff F is **line-perfect**, i.e., iff F does not contain any odd cycle of length at least 5 as a weak subgraph.

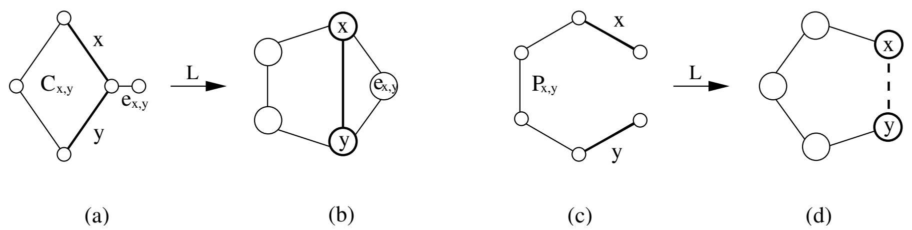

Figure 2: Definition of H-pairs and A-pairs.

In order to characterize critical and anticritical edges in $L(F)$, we define two structures in the underlying line-perfect graphs F. We say that two incident edges x and y form an **H-pair** in F if there is an edge $e_{x,y}$ incident to the common node of x and y and if there is a (not necessarily induced) even cycle $C_{x,y}$ containing x and y but only one endnode of $e_{x,y}$ (see Figure 2(a)). Then $L(C_{x,y})$ is an even hole and the node in $L(F)$ corresponding to $e_{x,y}$ has precisely two neighbors on $L(C_{x,y})$, namely x and y (see Figure 2(b)). Two

This paper appeared in: SIAM Journal on Computing 34:1 (2004), 109-117

non-incident edges x and y are called an **A-pair** if they are the endedges of a (not necessarily induced) odd path $P_{x,y}$ with length at least five (see Figure 2(c)). Then $L(P_{x,y})$ is an even, chordless path of length at least four with endnodes x and y or, in other words, $L(P_{x,y})$ is an odd hole where the edge between x and y is missing (see Figure 2(d)). It is straightforward that deleting and adding the edge xy in $L(C_{x,y} \cup e_{x,y})$ and $L(P_{x,y})$, respectively, yields an odd hole. In [12, 13] it is established that xy is a critical (anticritical) edge in $L(F)$ *only if* x, y form an H-pair (A-pair) in F.

Theorem 3 (WAGLER [12, 13]) *Let G be the line graph of a line-perfect graph F. An edge $e = xy$ of G (not contained in G) is critical (anticritical) if and only if x and y form an H-pair (A-pair) in F.*

Consequently, if $L(F)$ is intended to be (anti)critically perfect, *every* pair of (non-)incident edges in F must form an H-pair (A-pair). We call connected graphs with more than one edge **H-graphs** if each pair of incident edges forms an H-pair. (Disconnected critically perfect graphs only admit critically perfect components and can be created from connected critically perfect graphs by taking their disjoint union. Hence, it suffices to consider connected graphs.) We define a graph with at least two non-incident edges to be an **A-graph** if each pair of non-incident edges forms an A-pair. (This definition excludes only the cases that F is a triangle K_3 or a star $K_{1,k}$ for some $k \geq 1$.) It is an immediate consequence of Theorem 3 that F has to be a line-perfect H-graph (A-graph) if $L(F)$ is intended to be (anti)critically perfect. In [12, 13] it was shown that F must be bipartite in both cases.

Theorem 4 (WAGLER [12, 13]) *Let G be the line graph of F. G is critically (anticritically) perfect if and only if F is a bipartite H-graph (A-graph).*

Finding examples of critically (anticritically) perfect line graphs means, therefore, to look for bipartite H-graphs (A-graphs). The three smallest critically perfect graphs are the complements of the line graphs of the three bipartite A-graphs presented in Figure 3. A_1 is also an H-graph, hence $L(A_1)$ is bicritical (it is in fact self-complementary). All critically perfect graphs on 10 nodes are complements of line graphs of bipartite A-graphs. There are five simple A-graphs with 10 edges (see Figure 4). By definition, H-graphs have minimum degree 3. Hence none of the A-graphs in Figure 4 is an H-graph. The remaining five A-graphs with 10 edges arise from A_1, A_2, and A_3 by duplicating one edge. In general, duplicating edges preserves the property of being an A-graph.

This paper appeared in: SIAM Journal on Computing 34:1 (2004), 109-117

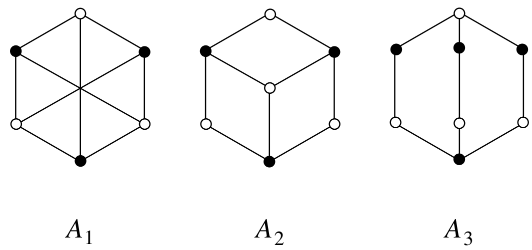

Figure 3: The three smallest bipartite A-graphs.

Lemma 5 (Wagler [13]) *If F is a bipartite A-graph and $ab \in E(F)$, then $F + ab$ is an A-graph.*

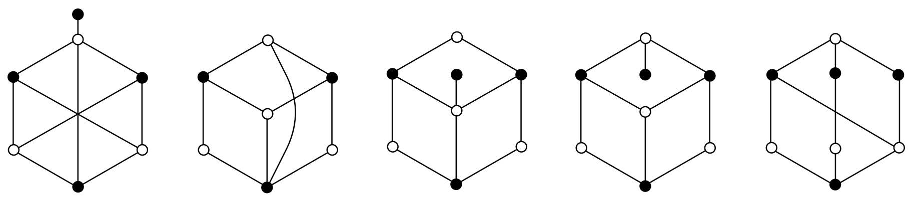

Figure 4: The simple, bipartite A-graphs with 10 edges.

This is not true, however, in the case of H-graphs:

Lemma 6 *If F is a bipartite H-graph and $ab \in E(F)$, then $F + ab$ is not an H-graph anymore.*

Proof. Consider a bipartite H-graph $F = (A \cup B, E)$ with color classes A, B and duplicate $e = ab \in E$ by $e' = ab$. Then the edges e and e' correspond in $L(F+e')$ to adjacent nodes v and v', respectively, with $N(v)-v' = N(v')-v$. That means, $L(F+e')$ arises from $L(F)$ by replacing the node v in $L(F)$ by the clique v, v'. Analogously, $L(F+e')-vv'$ can be generated from $L(F)$ by replacing the node v in $L(F)$ by the stable set v, v'. Due to the Replacement Lemma [9], $L(F+e')$ and $L(F+e')-vv'$ are perfect since $L(F)$ is, hence vv' cannot be a critical edge of $L(F+e')$. Theorem 4 implies that $F+e'$ is not an H-graph. $\square$.

Thus, none of the 10 A-graphs with 10 edges is an H-graph. This implies:

Corollary 7 *There is no bicritically perfect graph on 10 nodes.*

This paper appeared in: SIAM Journal on Computing 34:1 (2004), 109-117

It is not known so far whether there exist bicritically perfect graphs on 11 nodes. However, it is clear that there is no such *line* graph since the second smallest H-graph admits 12 edges. (A_1 is the only bipartite H-graph with 3 nodes in each color class. If there are 4 nodes in one color class, then an H-graph has at least 12 edges since it has minimum degree 3 by definition.) Finally, a sufficient condition for a bipartite graph F to be an H-graph *and* an A-graph is established in the next lemma, then $L(F)$ is bicritically perfect.

Lemma 8 (Wagler [13]) *Every simple, 3-connected, bipartite graph is an H-graph as well as an A-graph.*

3 Construction of the graphs G_n for $n \geq 12$

In order to treat Problem 1, this section is intended to present a bicritically perfect graph G_n for each $n \geq 12$. Lemma 8 ensures that $L(F)$ is bicritically perfect whenever F is a simple, 3-connected, bipartite graph. Hence we try to construct simple, 3-connected, bipartite graphs F_n with $n \geq 12$ edges to obtain the studied bicritically perfect graphs $G_n = L(F_n)$. Consider the bipartite graphs $F_{3k} = (A \cup B, E_1 \cup E_2)$ with $k \geq 3$ and

$$\begin{aligned} A &= \{1, 3, \ldots, 2k-1\} \\ B &= \{2, 4, \ldots, 2k\} \\ E_1 &= \{ii+1 : 1 \leq i \leq 2k \bmod(2k)\} \\ E_2 &= \{ii+3 : i \in A\} \end{aligned}$$

F_{3k} is an even cycle $(A \cup B, E_1)$ on its $2k$ nodes with k chords E_2 outgoing from the nodes in A with odd index. The three smallest examples of graphs F_{3k} for $k \in \{3, 4, 5\}$ are shown in Figure 5 (note $A_1 = F_9$).

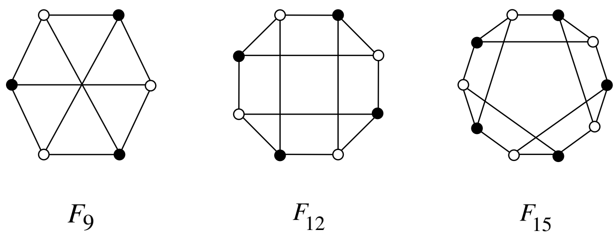

Figure 5: The graphs F_{3k} with $k = 3, 4, 5$.

Lemma 9 *The graphs F_{3k} are 3-connected for $k \geq 3$.*

This paper appeared in: SIAM Journal on Computing 34:1 (2004), 109-117

Proof. We have to show that the graph obtained from F_{3k} by removing two arbitrary nodes i and j is still connected. Let $i < j$. Recall that $F_{3k} = (A \cup B, E_1 \cup E_2)$ has a Hamilton cycle $C = (A \cup B, E_1)$ and additional chords $ii+3 \in E_2$ with $i \in A$ odd, $i+3 \in B$ even. If i and j are neighbors on C (i.e., $j = i+1$) then the remaining nodes $i+2 = j+1, \ldots, 2k, 1, \ldots, i-1$ of F_{3k} are connected by a path with edges in E_1. Otherwise, removing i and j divides the cycle C into two paths $P_1 = i+1, \ldots, j-1$ and $P_2 = j+1, \ldots, 2k, 1, \ldots, i-1$. It is easy to see that there is always an edge $e \in E_2$ which connects P_1 and P_2: If i is odd, then $i+1$ is even and $i-2i+1 \in E_2$. We have $e = i-2i+1$ as the studied edge connecting P_1 and P_2 if $i-2$ is a node of P_2 or else $V(P_2) = \{i-1\}$ and $j = i-2$ holds. But then $i-4$ is a node of P_1 (since $k \geq 3$) and we obtain $e = i-4i-1$ (all indices are taken modulo $2k$). Analogously, if i is even, then $i-1i+2 \in E_2$. We have $e = i-1i+2$ if $i+2$ is a node of P_1 or else $V(P_1) = \{i+1\}$ and $j = i+2$. But then $i+4$ is a node of P_2 (by $k \geq 3$ again) and we get $e = i+1i+4$. Hence, the graph obtained from F_{3k} by removing two arbitrary nodes is still connected. □

Thus, we can choose G_n as the line graph of F_n whenever $n = 3k, k \geq 3$ by Lemma 8 and Theorem 4. To close the gaps with $n = 3k+1, 3k+2$ for $k \geq 4$ we use the following immediate consequence of Lemma 8.

Lemma 10 *If $F = (A \cup B, E)$ is a simple, 3-connected, bipartite graph and $ab \notin E$ with $a \in A, b \in B$, then $F + ab$ is a bipartite A-graph and H-graph.*

Thus, we obtain the studied bipartite A- and H-graphs F_n for $n = 3k+1$ and $n = 3k+2$ if $k \geq 4$ by adding one and two edges, respectively, to F_{3k} such that the resulting graph is simple and bipartite. This is possible for each F_{3k} with $k \geq 4$ (but not for the complete bipartite graph F_9). We obtain, therefore, the studied bicritically perfect graphs $G_n = L(F_n)$ for $n \geq 12$ and can apply Strategy 1 for all cases with $n \geq 12$ nodes.

4 Construction of the graphs G_n for $n = 10, 11$

Due to Corollary 7 there is no bicritically perfect graph on 10 nodes. Moreover, it is unknown whether there is a bicritically perfect graph on 11 nodes but it is clear that there cannot be such a line graph. We try, therefore, to construct bipartite A-graphs with 10 and 11 edges which are closest to H-graphs as possible. Consider the bipartite A-graph $F_9 = A_1$. Duplicating an arbitrary edge of F_9 yields the graph F_{10} shown in Figure 6. F_{10} is an

This paper appeared in: SIAM Journal on Computing 34:1 (2004), 109-117

A-graph by Lemma 5 but not an H-graph due to Lemma 6. However, $L(F_{10})$ has only one non-critical edge, namely, the edge connecting the nodes that correspond to the parallel edges of F_{10}. Next, consider the bipartite graph F_{11} in Figure 6. It can be obtained by adding two edges to the A-graph A_2 from Figure 3. It is easy to check that F_{11} is an A-graph and that the two edges incident to the only node of degree two in F_{11} form the only non-H-pair in F_{11}. Theorem 3 implies that $L(F_{11})$ is anticritically perfect and has all but one critical edges, too.

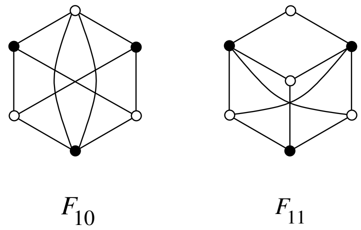

Figure 6: The graphs F_{10} and F_{11}.

Let us call a graph G **almost bicritically perfect** if either G is critically perfect and all but one edges not contained in G are anticritical or if G is anticritically perfect and all but one edges are critical. We call u and v the **indifferent pair** of an almost bicritically perfect graph if uv is its only non-critical or non-anticritical edge. Then we slightly modify Strategy 1 for almost bicritically perfect graphs as follows:

Strategy 2 Let G_n be an almost bicritically perfect graph on n nodes and let u, v be its indifferent pair.
Question 1:
Answer "$ij \in E$?" with YES. Number the nodes of G_n s.t. $i = u$, $j = v$.
For questions 2 to $\binom{n}{2} - 1$:
Answer "$ij \in E$?" with YES if $ij \in E(G_n)$, NO otherwise.

Then no imperfect subgraph appears during the first $\binom{n}{2} - 1$ questions, and the answer to the last question yields the decision again. Since $L(F_{10})$ and $L(F_{11})$ are almost bicritically perfect graphs by construction, we choose $G_{10} = L(F_{10})$ and $G_{11} = L(F_{11})$ and apply Strategy 2.

5 Concluding Remarks

In order to figure out whether perfectness is an elusive graph property, we used as main idea: Find, for as many numbers n of nodes as possible, a

This paper appeared in: SIAM Journal on Computing 34:1 (2004), 109-117

bicritically perfect graph G_n (Problem 1). Since one cannot reach another perfect graph from a bicritically perfect graph by deleting or adding one edge, there is a simple strategy for Player **B** in that case: Answer all but the last question as in the bicritically perfect graph (Strategy 1). Due to the enumeration of all critically perfect graphs on upto 10 nodes (Theorem 2) we pointed out that there is no bicritically perfect graph G_n with $n \leq 8$ and $n = 10$. With help of the characterization of critically and anticritically perfect line graphs (Section 2), we constructed bicritically perfect graphs G_n with $n = 9$ and $n \geq 12$ (Section 3) and almost bicritically perfect graphs G_{10} and G_{11} (Section 4) where a slightly different strategy has to be used (Strategy 2). Consequently, our main idea does not work for the small cases with $n \leq 8$ nodes but provides the desired result for all cases with $n \geq 9$ nodes. Hence, *perfectness is an elusive graph property for* $n \geq 9$ *nodes.*

Since the smallest imperfect graph is the odd hole C_5 on 5 nodes, perfectness is *not* elusive for $n \leq 4$ nodes: Player **A** knows without asking any question that the studied graph is perfect. Since the odd hole C_5 is the only imperfect graph on 5 nodes, one cannot reach another imperfect graph from the C_5 by deleting or adding one edge. Thus, the C_5 is bicritically imperfect and Strategy 1 does also work for $n = 5$ with choosing $G_n = C_5$. We are also able to settle the case $n = 6$ by providing some special strategy. For $n = 8$ we were able to prove elusiveness by making use of a result of Rivest and Vuillemin [10] and by doing some computer search. Details for the case $n = 6$ nd $n = 8$ are given in a forthcoming paper. There we also hope to settle the case $n = 7$ which is currently the only case for which it is not known whether perfectness is elusive.

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