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The P_4-structure of perfect graphs

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1 Introduction

One of the most important outstanding open problems in algorithmic graph theory is to determine the complexity of recognizing perfect graphs. Results of Lovász [33], Padberg [37] and Bland et al. [6] imply, as was first observed by Cameron [10] in 1982, that the problem of recognizing perfect graphs is in co-NP. So far it is not known whether this problem also belongs to NP, i.e., we do not know of any reasonable way to certify the perfection of an arbitrary graph.

One weak form of such a certificate is obtained via the Perfect Graph Theorem: to prove the perfection of a graph it is enough to show that its complement is perfect. In attempting to generalize this kind of certificate, Chvátal [11] invented in 1984 the notion of *P_4-structure*. For a given graph $G = (V, E)$, its P_4-structure is defined as the 4-uniform hypergraph on $V(G)$ whose edges are all the 4-element sets that induce a P_4 (i.e., a path on four vertices) in G.

Chvátal [11] conjectured that the perfection of a graph depends solely on its P_4-structure. He was led to this conjecture by observing that odd cycles and their complements have *unique* P_4-structure, i.e., any graph whose P_4-structure is isomorphic (as a hypergraph) to the P_4-structure of an odd cycle is an odd cycle or its complement. Therefore the truth of the Strong Perfect Graph Conjecture would imply his conjecture. Moreover, as the P_4 is a self-complementary graph, the P_4-structure of a graph and its complement are isomorphic. This shows that Chvátal's conjecture

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implies the Perfect Graph Theorem. Chvátal therefore suggested his conjecture be called the *Semi Strong Perfect Graph Conjecture*. In 1987, Reed [40] proved the conjecture and so it is now known as the *Semi Strong Perfect Graph Theorem*.

Meanwhile we have a fairly good knowledge of the P_4-structure of perfect and minimally imperfect graphs. It turns out that P_4-structure seems not to be powerful enough to get an NP-characterization for perfect graphs. Nevertheless the notion of P_4-structure has turned out to be a fruitful concept leading to many interesting structural and algorithmic results, not only in the area of perfect graphs.

In Section 2 we introduce the notion of P_4-structure and P_4-isomorphism and discuss the problem of recognizing the P_4-structures of graphs which is the central problem in this area. Section 3 introduces the notions of modules and homogeneous sets and their counterparts in hypergraphs called h-sets. The powerful tools of modular and homogeneous decompositions allow the efficient solution of the recognition problem and some optimization problems for several classes of graphs. Moreover, they suggest a technique for solving the P_4-structure recognition problem and indicate why the P_4-structures of split graphs are of special interest in this context. Section 4 discusses the Semi Strong Perfect Graph Theorem, its consequences for the problem of recognizing perfect graphs, and its connection to the problem of recognizing the P_4-structures of graphs. In Section 5 we study graphs that have unique respectively strongly unique P_4-structure. Based on the results in this section a polynomial time algorithm to recognize P_4-structures of graphs is outlined in Section 6. Also a survey on the recognition of the P_4-structure of special classes of (perfect) graphs is given in this section. Section 7 surveys results on the P_4-structure of minimally imperfect graphs. Section 8 contains some extensions of the notion of P_4-structure and Section 9 shows that several results proven for P_4-structure also hold for P_3-structure.

2 P_4-Structure: Basics, Isomorphisms and Recognition

The *P_4-structure* of a graph $G = (V, E)$ is a 4-uniform hypergraph on $V(G)$ whose hyperedges are all sets on four vertices that induce a P_4 (a path on four vertices) in G. Figure 1 shows an example of a graph and its P_4-structure. Note that the P_4-structure of a graph contains only the information which vertices in the graph induce a P_4, but no information how these four vertices are connected to induce the P_4. If for a 4-uniform hypergraph $\mathcal{H}$ there exists a graph G such that $\mathcal{H}$ is the P_4-structure of G, then we call G a *realization* of $\mathcal{H}$. We say that a hypergraph $\mathcal{H}$ has a *unique realization*, if it has exactly one realization or if it has exactly two realizations which

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are complements of each other.

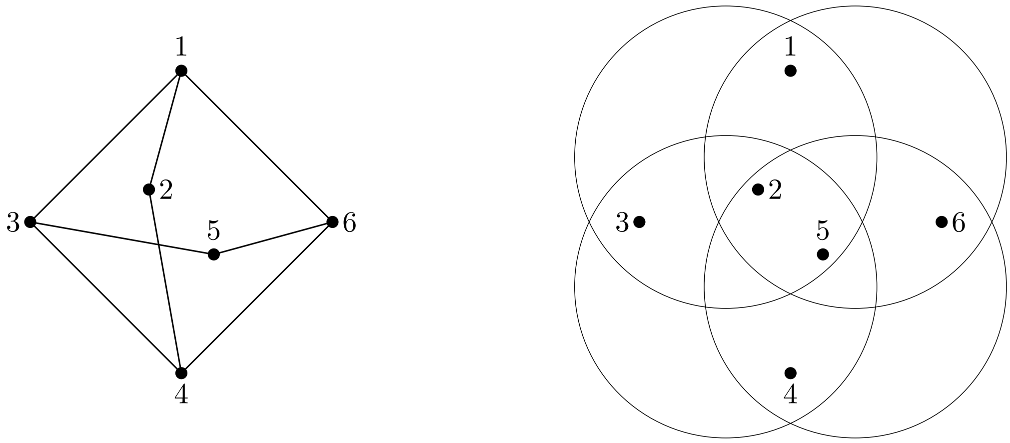

Figure 1: A graph and its P_4-structure.

Two graphs are said to have the *same P_4-structure* if their P_4-structures are isomorphic (as hypergraphs). To test whether two given graphs G and H have the same P_4-structure it is not necessary to first compute their P_4-structure and then to decide whether these two hypergraphs are isomorphic. As an immediate consequence from the definition of P_4-structure the following proposition shows a simpler way to check whether two graphs have the same P_4-structure.

Proposition 1 *Two graphs G and H have the* same P_4-structure *if there exists a bijection $\varphi : V(G) \to V(H)$ such that four vertices $\{a, b, c, d\}$ induce a P_4 in G if and only if $\{\varphi(a), \varphi(b), \varphi(c), \varphi(d)\}$ induces a P_4 in H.* □

The bijection φ used in Proposition 1 is called a *P_4-isomorphism* between G and H. Figure 2 shows an example of four different graphs that all have the same P_4-structure. It is easily verified that four vertices induce a P_4 in one of these graphs if and only if the four vertices with the same labels induce a P_4 in the other three graphs.

Clearly, the relation "having the same P_4-structure" defines an equivalence relation on the set of all graphs. The equivalence classes defined this way are called the *P_4-isomorphism classes.* The question arises, how difficult it is to decide whether two graphs belong to the same P_4-isomorphism class, i.e., what is the complexity of the following decision problem:

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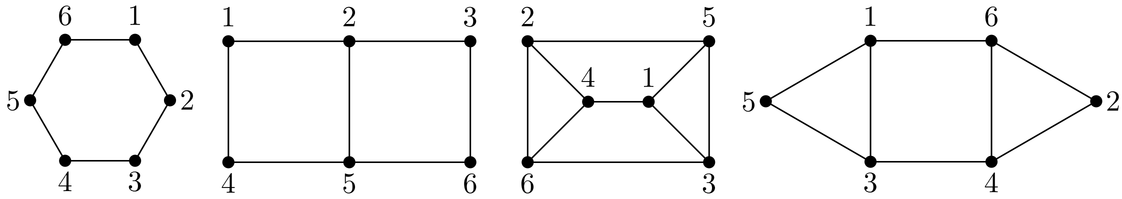

Figure 2: Example of graphs having the same P_4-structure.

PROBLEM	P_4-ISOMORPHISM
Input:	Two graphs G and H
Question:	Do G and H have the same P_4-structure ?

This problem is closely related to the problem GRAPHISOMORPHISM, i.e., the problem of deciding whether two given graphs are isomorphic. This problem is one of the most studied problems that has neither been shown to be in P, nor been shown to be NP-complete. See [39], [22] and [31] for more background on the graph isomorphism problem and [7], [42] for arguments why it is unlikely that GRAPHISOMORPHISM is an NP-complete problem.

Reed [41] provided a simple reduction that shows that the problem P_4-ISOMORPHISM is polynomial time equivalent to GRAPHISOMORPHISM, i.e., a polynomial time algorithm for one of the two problems would also yield a polynomial time algorithm for the other problem.

Lemma 1 (Reed 1986 [41])
P_4-ISOMORPHISM *is polynomial time equivalent to* GRAPHISOMORPHISM.

Sketch of proof: To transform an instance of P_4-ISOMORPHISM into an instance of GRAPHISOMORPHISM construct for a given graph G a bipartite graph as follows: vertices of the new graph are all vertices of G together with all induced P_4's of G and two additional vertices x and y. Now connect in the new graph each vertex of G with all P_4's that contain this vertex and connect all P_4's with x and add the edge xy. It is easily seen that two graphs are P_4-isomorphic if and only if the bipartite graphs constructed this way are isomorphic as graphs.

To transform an instance of GRAPHISOMORPHISM into an instance of P_4-ISOMORPHISM it is enough to add for every edge in a given graph two cycles of length at least $|G|+1$ that are identified in the edge of the graph. The correctness of this transformation follows from Lemma 4. □

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One of the central algorithmic questions concerning P_4-structure is whether one can efficiently decide for a given 4-uniform hypergraph whether there exists a graph that has this hypergraph as its P_4-structure.

PROBLEM	P_4-StructureRecognition
Input:	A 4-uniform hypergraph $\mathcal{H}$
Question:	Is there a graph having $\mathcal{H}$ as its P_4-structure ?

The problem P_4-StructureRecognition and its solution will be studied in more detail in Section 3 and Section 6. Its connection with the problem of recognizing perfect graphs is discussed in Section 4.

3 Modules, h-Sets, Split Graphs and Unique P_4-Structure

For many P_4-structures $\mathcal{H}$, we can efficiently find, not just one realization of $\mathcal{H}$ but all of its realizations. Indeed, as we shall see, it is the fact that we can often solve this more difficult problem which allows us to obtain a recursive algorithm to solve P_4-StructureRecognition. We cannot hope to solve this problem efficiently for all P_4-structures as some P_4-structures have exponentially many realizations. One important part of our approach is to analyze the structure of P_4-structures with many realizations. We shall be particularly interested in characterizing graphs with unique P_4-structures. We begin our discussion in this section by studying the relationship between unique P_4-structures and *modules*.

A *module* M in a graph G is a set of vertices such that no vertex outside M can distinguish between the vertices in M, i.e., every vertex outside of M is either adjacent to all or no vertices in M. A module is called *trivial* if it contains only one or all vertices of G. A nontrivial module is called a *homogeneous set*. See Figure 3 for two examples of homogeneous sets. A graph without a homogeneous set is called *prime*.

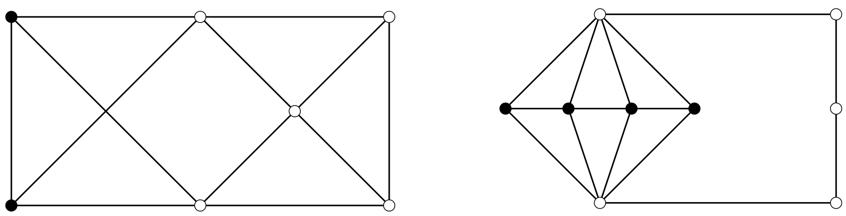

Figure 3: Graphs with homogeneous set (bold vertices).

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There exists a close relationship between the modules of a graph and its P_4-structure because no P_4 contains a non-trivial module. Thus, a P_4 intersects a module in 0,1, or 4 vertices. More strongly we have the following:

Key Fact: *For any module M, every P_4 is either contained in M, disjoint from M, or has exactly one vertex in M. Furthermore, if some P_4 P intersects a module M in exactly one vertex x then $P - x + y$ is a P_4 for every vertex y in M.*

This is the easy direction of a classical result of Seinsche concerning graphs whose P_4-structures have no edges, the so-called *P_4-free* graphs.

Lemma 2 (Seinsche 1986 [43])
A graph is P_4-free if and only if every induced subgraph with at least three vertices contains a homogeneous set. □

Our Key Fact also has implications concerning the uniqueness of non-empty P_4-structures. The graph G_2 in Figure 4 is obtained from the graph G_1 by replacing the subgraph induced by the vertices of M by its complement. We note that these two graphs have the same P_4-structure. In fact any two graphs one of which is obtained from the other by replacing the graph induced by a module by its complement will have the same P_4-structure. In general, these two graphs will not be isomorphic and therefore neither will have unique P_4-structure (there are exceptions – see Figure 8). It is this fact which motivates our study of modules.

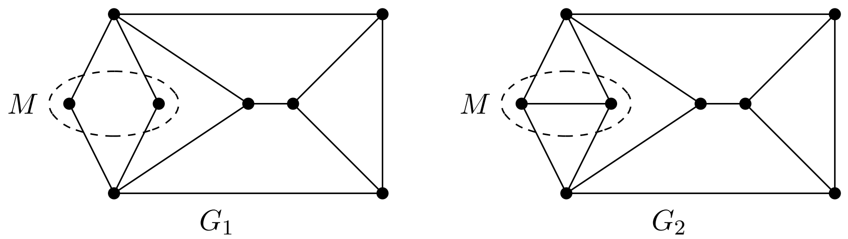

Figure 4: Replacing a module M by its complement.

The following result, which can be found in Chapter **** of this book, is the foundation on which the study of modules is built:

Theorem 1 (Modular Decomposition Theorem, Gallai 1967 [21])
For any graph $G = (V, E)$ with at least two vertices precisely one of the following conditions holds:

(i) G is disconnected

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(ii) $\overline{G}$ *is disconnected*

(iii) There exists some $Y \subseteq V, |Y| \geq 4$ *and a unique partition* $V_1, V_2, \ldots$ *of* V *such that* Y *induces a maximal prime subgraph in* G *and every* V_i *is a module with* $|V_i \cap Y| = 1$.

A modular decomposition of any graph can be built up by recursively applying this theorem. See Figure 5 for an example of a modular decomposition tree. Formally, the modular decomposition of G is a tree-representation of all the (possibly exponentially many) modules of G that requires only linear space. (It is the transitive reduction of the containment relation on the set of all modules which is unique up to tree isomorphisms.)

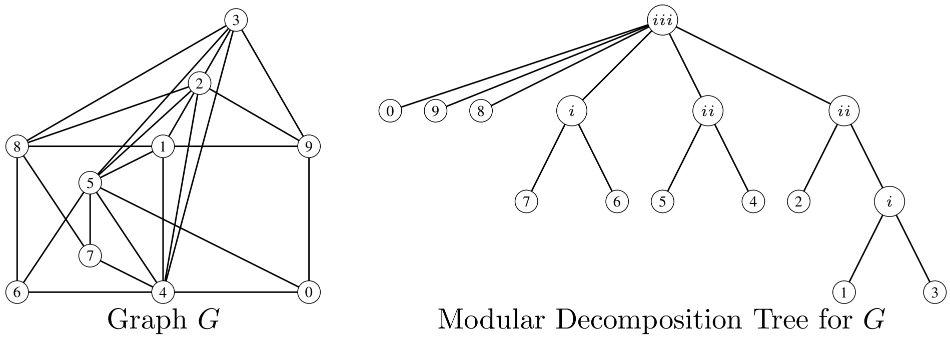

Figure 5: The modular decomposition tree of a graph G. The nontrivial modules in this graph are the sets $\{1, 2, 3\}, \{1, 3\}, \{4, 5\}$ and $\{6, 7\}$. The labels at the internal nodes of the decomposition tree indicate which operation of the Modular Decomposition Theorem was applied.

The idea of modular decomposition goes back to Gallai [21] and is also known as *substitution decomposition* [35], *prime tree decomposition* [20] and *X-join decomposition* [23] and has been extended to more general combinatorial objects.

The knowledge of the modular decomposition of a graph allows in several cases the efficient solution of optimization problems or the recognition problem for the underlying class of graphs (see for example [45]). It is therefore desirable to have very efficient algorithms to compute modular decompositions of graphs. In case of P_4-free graphs a linear time algorithm to compute the modular decomposition was given by Corneil, Perl and Stewart in 1985 [15]. Recently McConnell and Spinrad [34] found such an algorithm for general graphs. Their algorithm also implies a linear time recognition method for comparability graphs. Another linear time algorithm to compute the modular decomposition of graphs was presented by Cournier and

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Habib [16]. Their algorithm is applicable to directed and undirected graphs. Efficient parallel algorithms and related algorithmic results for computing the modular decomposition can be found in [17] [18] [32].

The close relationship between modules and P_4's discussed above motivates our definition of an analogous notion for P_4-structures. A nontrivial subset S of the vertices of a 4-uniform hypergraph $\mathcal{H}$ is called an *h-set*, if every hyperedge in $\mathcal{H}$ intersects S in $0, 1$ or 4 vertices. Furthermore, if a hyperedge h intersects S in exactly one vertex x then $h - x + y$ is a hyperedge in $\mathcal{H}$ for every y in S. Obviously, if a graph contains a homogeneous set, then its P_4-structure contains an h-set. The converse is not necessarily true as shown by an example in Figure 6. Both graphs shown in this figure have the same P_4-structure which contains an h-set, but only the graph depicted on the right contains a homogeneous set.

Figure 6: Two graphs with h-sets in the P_4-structure.

The graph that is obtained from a given graph by shrinking all maximal homogeneous sets into a single vertex is called its *characteristic graph*. Similarly, the hypergraph obtained by shrinking all maximal h-sets is called the *characteristic hypergraph*. Given a vertex x in a graph G_1 and some other graph G_2 one can *substitute* x *by* G_2 which means the following: Form the disjoint union of $G_1 - x$ and G_2 and connect every vertex of G_2 with all neighbors of x in G_1. Clearly, in this new graph the graph G_2 is a module. The following key result is an immediate consequence from the definition of h-sets and our Key Fact:

Proposition 2 *Let S be an h-set in a 4-uniform hypergraph $\mathcal{H}$. Let x be a vertex of S. Suppose G_1 is a realization of the subhypergraph induced by $\mathcal{H} - (S - x)$ and G_2 is a realization of the subhypergraph induced by S. Then substituting G_2 for x in G_1 yields a realization of G.* $\square$

The above remarks suggest that we should be particularly interested in hypergraphs which contain no h-sets. We call such hypergraphs *prime*.

An h-set in a 4-uniform hypergraph $\mathcal{H}$ can easily be found in polynomial time by making use of the following algorithm [26]:

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ALGORITHM FIND h-SET $(a,\, b)$

Input: A 4-uniform hypergraph $\mathcal{H}$ and two vertices a and b
Output: An h-set containing a and b if exists.

S := {a, b}
while $\exists f \in E(\mathcal{H})$ with $|S \cap f| \in \{2, 3\}$ **or**
$S \cap f = \{s\}$ and $\exists x \in S$ s. t. $f - s + x \notin E(\mathcal{H})$ **do**
add f to S

if $S \neq V(\mathcal{H})$ **then** output S
else no h-set in $\mathcal{H}$ contains a and b

Now, we can apply the algorithm above to determine if $\mathcal{H}$ has an h-set and if so apply Proposition 2 to reduce our problem to two subproblems. Repeating this process allows us to restrict our attention to prime hypergraphs.

Unfortunately, there are also prime hypergraphs which do not have unique realizations. For one example, see Figure 7. The graphs in Figure 7 are *split graphs*, that is the vertices can be partitioned into two sets such that one of the two sets induces a clique and the other a stable set. Note that this implies that every P_4 has its endpoints in the stable set and its midpoints in the clique. It follows easily that replacing the clique by a stable set and the stable set by a clique yields a new split graph with the same P_4-structure. The next decomposition allows us to handle this and similar situations.

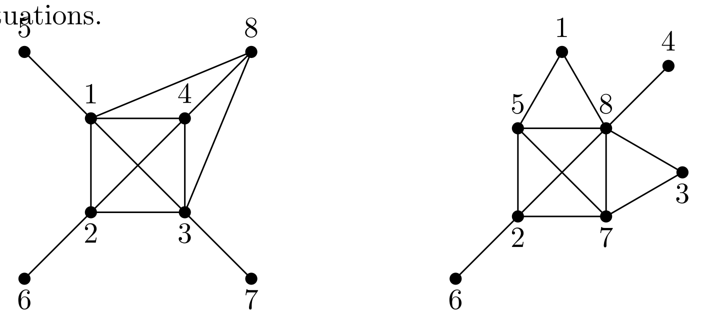

Figure 7: Two prime graphs with the same P_4-structure which are not complements of each other.

The modular decomposition has been extended by Jamison and Olariu [30] to the so called *homogeneous decomposition*. It gives a refined decomposition of graphs by permitting the decomposition of graphs that are prime with respect to the modular decomposition. As with the modular decomposition, the homogeneous decomposition of a graph can be computed in linear time [5] and is therefore the basis for several efficient optimization algorithms, see [4] for a survey of results of this type.

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The basis for homogeneous decomposition is the notion of *p-connectedness*. A graph is called p-connected, if for every partition of its vertex set into two nonempty disjoint sets, there exists a P_4 in the graph that intersects both sets. A maximal p-connected subgraph is called a *p-component*. A p-connected graph is called *separable* if its vertex set can be partitioned into two nonempty sets such that each P_4 that is not completely contained in one of the two sets has its endpoints in one set and its midpoints in the other set. The foundation of the homogeneous decomposition of graphs is given by the following structure theorem:

Theorem 2 (Primeval Decomposition Theorem, Jamison and Olariu 1995 [30])

For any graph $G = (V, E)$ precisely one of the following conditions holds:

(i) *G is disconnected*

(ii) *$\overline{G}$ is disconnected*

(iii) *G is p-connected*

(iv) *There is a unique proper separable p-component H of G with a partition (H_1, H_2) such that every vertex outside H is adjacent to all vertices in H_1 and misses all vertices in H_2.*

By recursively applying the Primeval Decomposition Theorem one gets the so called *primeval decomposition tree* of a graph. The homogeneous decomposition of a graph is obtained by using two more decomposition operations: one operation that allows substituting vertices by homogeneous sets and another one that gives a clique representation for split graphs. Of special interest in this context is the following result of Jamison and Olariu that shows the significance of split graphs for the problem P_4-StructureRecognition:

Lemma 3 (Jamison, Olariu 1995 [30])
A p-connected graph is separable if and only if its characteristic graph induces a split graph. □

4 The Semi Strong Perfect Graph Theorem and Certifying Perfection

The Perfect Graph Theorem induces an equivalence relation on the set of all graphs by putting two graphs into the same equivalence class, if they are complements of

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each other. The statement of the Perfect Graph Theorem then simply is that if we partition the set of all graphs into perfect and imperfect graphs, then any such equivalence class will completely belong to one side of the partition. As the P_4 is a selfcomplementary graph, this implies that the P_4-isomorphism classes extend these equivalence classes. Chvátal [11] conjectured that the P_4-isomorphism classes also have the property that their members are either all perfect or all are imperfect. This conjecture was called the *Semi Strong Perfect Graph Conjecture*. Chvátal was led to this conjecture by observing that odd cycles of length at least five and their complements have *unique P_4-structure*, i.e., the P_4-isomorphism class of these graphs contain only the graph and its complement. Hayward then observed [24] that the same also holds for long enough even cycles, so we have:

Lemma 4 (Chvátal 1984 [11], Hayward 1983 [24])
Odd cycles of length at least five and even cycles of length at least 8 have unique P_4-structure. □

Note that the C_6 does not have unique P_4-structure as all graphs in Figure 2 have the P_4-structure of a C_6.

The Strong Perfect Graph Conjecture together with Lemma 4 implies the truth of the Semi Strong Perfect Graph Conjecture. On the other hand the Perfect Graph Theorem is implied by the Semi Strong Perfect Graph Conjecture, which explains why it was given this name.

The Semi Strong Perfect Graph Conjecture first has been proved for the special cases of bipartite graphs [14] and triangulated graphs [24] until in 1987 Reed gave a proof for the conjecture in general which was from then on called the *Semi Strong Perfect Graph Theorem.*

Theorem 3 (The Semi Strong Perfect Graph Theorem, Reed 1987 [40])
A graph is perfect if and only if it has the P_4-structure of a perfect graph. □

This result shows that for solving the recognition problem for perfect graphs, i.e. the problem

PROBLEM	PERFECTGRAPHRECOGNITION
Input:	A graph G
Question:	Is G perfect ?

it suffices to be able to identify P_4-structures that belong to perfect graphs. This problem is called PERFECTP_4-STRUCTURERECOGNITION:

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PROBLEM	PERFECTP_4-STRUCTURERECOGNITION
Input:	A 4-uniform hypergraph $\mathcal{H}$
Question:	Is there a perfect graph having $\mathcal{H}$ as its P_4-structure ?

Chvátal[13] asked whether there exists a polynomial time algorithm for the problem PERFECTP_4-STRUCTURERECOGNITION. Clearly, such an algorithm would yield a polynomial time algorithm for the problem of recognizing perfect graphs. Therefore, Chvátal also asked for a simpler problem, namely the recognition of P_4-structures for special classes of (perfect) graphs. Such a result could lead to an NP characterization of perfect graphs as follows. To certify the perfection of a given graph G it is enough to present a graph H for which its perfectness is easily verified and that has the same P_4-structure as G. The question of course arises whether such a graph H always exists. There are many classes of graphs for which we know the perfectness as for example bipartite graphs or triangulated graphs. Therefore a natural question is whether there exists a simple class $\mathcal{C}$ of perfect graphs that intersects every P_4-isomorphism class that contains only perfect graphs.

Question 1 *Is there some simple class $\mathcal{C}$ of perfect graphs such that for each perfect graph G there is a graph in $\mathcal{C}$ that has the same P_4-structure as G ?*

Clearly, a positive answer to Question 1 would imply that the problem PERFECTGRAPHRECOGNITION belongs to NP. We may assume that a class $\mathcal{C}$ that could be an answer to Question 1 is closed under taking graph complements, as these can be computed in polynomial time. Therefore, a necessary condition for a class $\mathcal{C}$ is that it must contain all graphs that have unique P_4-structure. This leads us to the next question:

Question 2 *Which (perfect) graphs have unique P_4-structure ?*

We have already touched on this question in Section 3, we will discuss it further in the next section. This discussion implies that we get a negative answer to Question 1.

5 The Structure of the P_4-Isomorphism Classes

To answer Question 2 we need to know which (perfect) graphs have unique P_4-structure. First we will mainly be interested in asymptotic results which will tell us how many (perfect) graphs have unique P_4-structure. For a class $\mathcal{C}$ of graphs and a graph property Π we say that *almost all graphs in $\mathcal{C}$ have property* Π if the ratio $|\mathcal{C}_n(\Pi)|/|\mathcal{C}_n|$ tends to 1, as n goes to infinity. Here $\mathcal{C}_n$ denotes all graphs in $\mathcal{C}$ on n vertices and $\mathcal{C}_n(\Pi)$ denotes all graphs in $\mathcal{C}_n$ that have property Π.

It is easily seen that the P_4-isomorphism classes can be arbitrarily large, as for

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example all P_4-free graphs belong to the same P_4-isomorphism class. Thus, the following result may be surprising at the first sight, but a close look at Reed's proof of the Semi Strong Perfect Graph Theorem [40] reveals that a weaker form of the following result is essentially the idea of the proof.

Theorem 4 (Hougardy 1996 [28])
Almost all graphs have unique P_4-structure. □

This result shows that the P_4-isomorphism classes almost always contain just two elements, namely the graph and its complement. But, as almost no graph is perfect, it could still be the case that on the class of perfect graphs these equivalence classes have almost always a much larger size. The following result shows that this is not the case.

Theorem 5 (Hougardy 1996 [28])
Almost all perfect graphs have unique P_4-structure. □

The proof of this result is based on a result of Prömel and Steger [38] who proved that almost all Berge graphs are perfect, i.e., they showed that the Strong Perfect Graph Conjecture is almost always true.

As noted in Section 3 there exists a close relation between the P_4-structure of graphs and homogeneous sets.

As we saw in Section 3, most graphs containing a homogeneous set do not have unique P_4-structure. Some exceptions are shown in Figure 8 We need a slightly more restricted notion of unique P_4-structure to get the desired connection between homogeneous sets and unique P_4-structure. A graph is said to have *strongly unique P_4-structure* if it has unique P_4-structure and every P_4-automorphism is also a graph automorphism.

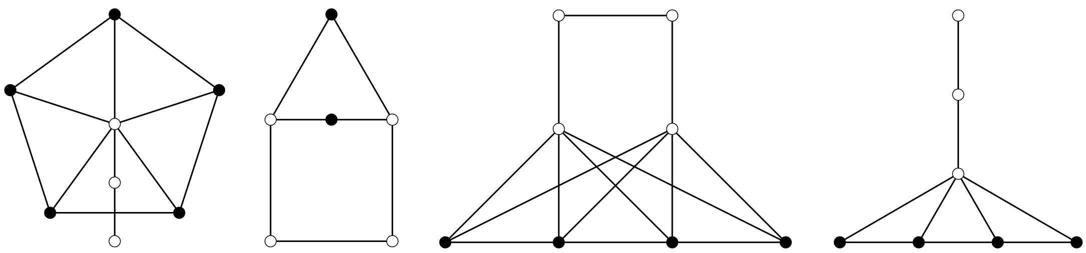

Figure 8: Graphs with homogeneous set (bold vertices) and unique P_4-structure.

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Clearly, if a graph has strongly unique P_4-structure it also has unique P_4-structure. The C_5 is an example of a graph that has unique but not strongly unique P_4-structure: each of the 120 permutations of the five vertices is a P_4 isomorphism, but only 10 of them are graph isomorphisms.

If a graph G has a homogeneous set S and we take the complement on the graph induced by S, this will give us a P_4-isomorphism but not a graph isomorphism or an isomorphism to the complement of G. Thus, if a graph has a homogeneous set it cannot have strongly unique P_4-structure. The other direction does not hold in general, as for example the P_5 or C_6 provide examples of graphs without homogeneous set which do not have strongly unique P_4-structure. But it turns out, that graphs without homogeneous set that contain a certain fixed induced subgraph all have strongly unique P_4-structure.

Theorem 6 (Hougardy 1996 [28])
If a graph G contains an induced cycle of length at least seven then G has strongly unique P_4-structure if and only if G does not contain a homogeneous set. □

Theorem 6 was generalized by Hayward, Hougardy and Reed, as follows. A graph is *CSC* if it can be partitioned into two cliques with no edges between them and a stable set. It is *SCS* if its complement is CSC.

Theorem 7 (Hayward, Hougardy, Reed 1996 [26])
If a prime graph G contains a (prime) subgraph F which has a strongly unique P_4-structure then provided F is neither SCS nor CSC, G also has a strongly unique P_4-structure.

The example of Figure 9 shows that the condition that F is not SCS or CSC is necessary. All the graphs of Figure 10 are neither SCS nor CSC and all have strongly unique P_4-structure. Thus Theorem 6 remains true if we replace cycle of length seven by any graph in Figure 10.

Theorem 7 has negative consequences for the applicability of P_4-structure to certify perfection: for all perfect graphs that contain this subgraph, P_4-structure is no better than homogeneous sets in certifying perfection. Theorem 6 and 7 show that the class $\mathcal{C}$ that was asked for in Question 1 cannot exist. On the other hand as we see in the next section, Theorem 7 is very helpful in designing an algorithm to solve P_4-StructureRecognition.

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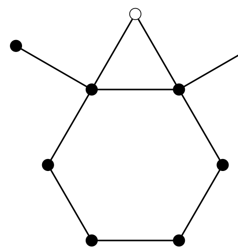

Figure 9: A prime graph without strongly unique P_4-structure even though the subgraph induced by the bold vertices has strongly unique P_4-structure.

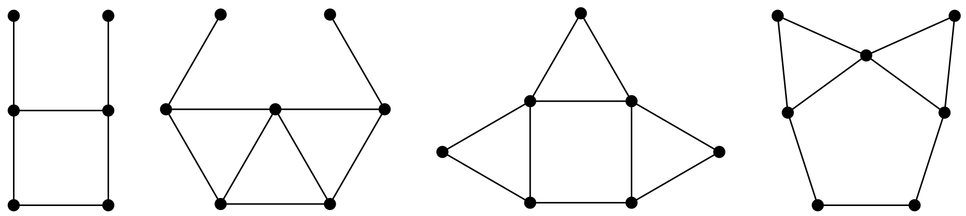

Figure 10: Graphs with strongly unique P_4-structure that are neither SCS nor CSC.

6 Recognizing P_4-Structure

The preceding sections have shown how the problem P_4-StructureRecognition relates to the problem PerfectGraphRecognition. While the latter problem is still open, the problem P_4-StructureRecognition allows a polynomial time algorithm for its solution as was shown by Hayward, Hougardy and Reed:

Theorem 8 (Hayward, Hougardy, Reed 1996 [26])
The problem P_4-StructureRecognition can be solved in polynomial time. □

Theorem 7 is the key to the algorithm of Hayward, Hougardy and Reed. They also need the following results:

Lemma 5 *If $\mathcal{F}$ is a prime hypergraph of a prime P_4-structure $\mathcal{H}$ then there is a prime subhypergraph of $\mathcal{H}$ containing $\mathcal{F}$ and at most two other vertices.* □

This allows them to solve P_4-StructureRecognition, given a strongly unique realization of a subhypergraph $\mathcal{F}$ of $\mathcal{H}$ which is neither SCS or CSC as follows:

They repeatedly find a larger prime hypergraph by applying Lemma 5 and find the corresponding unique realization (it turns out to be fairly straightforward to find all the 2 vertex extensions of a given realization).

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The rest of the algorithm involves dealing with hypergraphs for which we cannot find such a subhypergraph $\mathcal{F}$. One crucial point is to have a subroutine that checks whether a 4-uniform hypergraph $\mathcal{H}$ has a realization as a split graph. Hayward, Hougardy and Reed [26] present a polynomial time algorithm for this problem restricted to h-set free hypergraphs and show how it can be extended to hypergraphs containing h-sets. Their algorithm starts with the realization of one single hyperedge partitioned into four sets W, X, Y, Z and successively adds in each step one or two vertices to the sets W, X, Y, Z such that the following two conditions hold

- In any realization of $\mathcal{H}$ as a split graph the vertices from the same set must belong to the same side of the split graph
- There exists a P_4 with a vertex in each of W, X, Y, Z.

When posing the problem P_4-StructureRecognition Chvátal [13] also asked whether this problem can be solved for some interesting subclasses of (perfect) graphs.

PROBLEM	P_4-StructureRecognitionForClassC
Input:	A 4-uniform hypergraph $\mathcal{H}$
Question:	Is there a graph in $\mathcal{C}$ having $\mathcal{H}$ as its P_4-structure ?

The first class $\mathcal{C}$ for which a polynomial time algorithm for the problem P_4-StructureRecognitionForClassC was presented was the class of trees. Ding [19] gave such an algorithm whose running time was later improved by Brandstädt, Le and Olariu [9]. Meanwhile the problem P_4-StructureRecognitionForClassC has been proved for many other classes of (perfect) graphs. For example Babel, Brandstädt and Le solved the problem for bipartite graphs [3], Brandstädt and Le for block graphs [8], Sorg for linegraphs of bipartite graphs [44], Babel for linegraphs [1] and for claw-free graphs [2], and Hayward, Hougardy and Reed for split graphs [26].

7 The P_4-Structure of Minimally Imperfect Graphs

A graph is called *unbreakable* if neither the graph nor its complement contains a star-cutset. Chvátal's Star-Cutset Lemma states that minimally imperfect graphs are unbreakable. To prove the Strong Perfect Graph Conjecture it is therefore enough to have it proved for the class of unbreakable graphs. This explains why studying the P_4-structure of unbreakable graphs is of special interest with respect to perfect

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graphs. There exist several results that show that the P_4-structure of an unbreakable graph is in some sense densely connected.

Two P_4's h and h' in a graph G are *3-chained* if there exists a sequence $h = h_1, h_2, \ldots, h_k = h'$ of P_4's in G such that any two consecutive ones have exactly three vertices in common.

Lemma 6 (Chvátal 1987 [12])
In an unbreakable graph every two P_4's are 3-chained. □

An *alignment* in a graph is a sequence $Q_1, Q_2, \ldots, Q_k$ of sets of vertices such that each Q_i induces a P_4, and each Q_i with $i \geq 2$ has precisely one vertex outside $Q_1 \cup Q_2 \cup \ldots \cup Q_{i-1}$. The alignment is called *full* if each vertex of the graph belongs to at least one Q_i.

Lemma 7 (Chvátal, Hoàng 1985 [14])
In an unbreakable graph every alignment extends into a full alignment. □

Both these results have been extended by Olariu [36]. He partitions the edges of a graph G into *coercion classes* such that two edges xy and ab belong to the same coercion class if $abxy$ is a P_4 in G.

Theorem 9 (Olariu 1991 [36])
An unbreakable graph contains at most two coercion classes and in an unbreakable graph with exactly two coercion classes the neighborhood of any vertex is the disjoint union of two cliques. □

Any counterexample to the Strong Perfect Graph Conjecture must be C_5-free and unbreakable which makes this class of graphs of special interest. For the class of C_5-free unbreakable graphs we can give a complete description of the structure of the P_4-isomorphism classes.

Theorem 10 (Hougardy 1997 [29])
C_5-free unbreakable graphs different from C_6 and its complement have unique P_4-structure. □

This result shows that for the class of C_5-free unbreakable graphs the Semi Strong Perfect Graph Theorem and the Perfect Graph Theorem make exactly the same statement.

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Theorem 10 cannot be extended to the class of all unbreakable graphs as there exist unbreakable graphs different from C_6 and its complement that have the same P_4-structure. Figure 11 shows two such graphs.

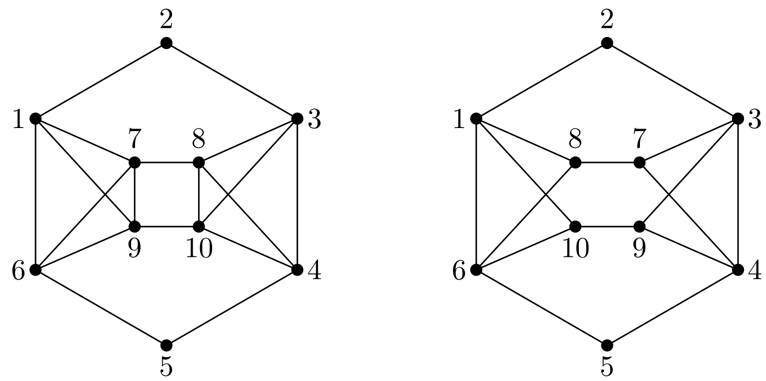

Figure 11: Two unbreakable graphs with the same P_4-structure

From Theorem 10 one can easily derive the Semi Strong Perfect Graph Theorem. However, it should be noted, that the proof of Theorem 10 is longer than the proof of the Semi Strong Perfect Graph Theorem, thus the following proof is not really a better proof to the Semi Strong Perfect Graph Theorem but sheds some more light on it.

Short proof for the SSPGT
Assume G and H are two graphs having the same P_4-structure such that G is perfect and H is minimally imperfect. Chvátal's Star-Cutset Lemma implies that H is C_5-free and unbreakable. Now Theorem 10 shows that H has unique P_4-structure, i.e. G or its complement must be isomorphic to H, which is a contradiction. □

8 The Partner Structure and Other Generalizations

We have seen in Section 5, that the P_4-isomorphism classes almost always are of size at most two. Thus, a natural question arises whether the notion of P_4-structure can be extended in such a way that the equivalence classes defined by this extension become larger and still a statement similar to the Semi Strong Perfect Graph Theorem holds. One such approach uses the notion of partner structure.

Two vertices a and b of a graph G are called *partners* if there are vertices x, y, z in $G - \{a, b\}$ such that $\{a, x, y, z\}$ and $\{b, x, y, z\}$ each induce a P_4 in G. The *partner*

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graph of a graph G is the graph whose vertices are the vertices of G, and whose edges are the pairs of vertices that are partners in G. The notion of partner graphs was introduced by Chvátal [12] who proved the following beautiful decomposition result:

Lemma 8 (Chvátal [12] 1987)
If the vertices of a graph G can be colored by two colors such that any two partners obtain the same color, then G is perfect if and only if the two graphs induced by the two color-classes are perfect.

To test whether a given graph admits a two-coloring as requested in the lemma, one simply has to check whether its partner-graph is connected. Therefore a decomposition according to Chvátal's result can be found in polynomial time. However, not all perfect graphs admit such a decomposition.

Motivated by Chvátal's result Hoàng [27] defined two graphs G and G' to have the same *partner-structure* if there exists a mapping $f : V \to V'$ such that vertices x, y are partners with respect to a set $\{a, b, c\}$ in G if and only if $f(x), f(y)$ are partners with respect to $f(\{a, b, c\})$ in G'. Thus, the partner-structure of a graph is obtained by removing all hyperedges from its P_4-structure that do not intersect some other hyperedge in exactly three vertices. Obviously, if two graphs have the same P_4-structure then they also have the same partner-structure. The other direction does not hold as can be seen for example by considering the P_4 and the K_4. Thus, the following result of Hoàng is a generalization of the Semi Strong Perfect Graph Theorem. Its relation to the Semi Strong Perfect Graph Theorem is studied in [28].

Theorem 11 (Hoàng 1990 [27])
Let G and G' be two graphs with the same partner-structure. Then G is perfect if and only if G' is. □

9 P_3-Structure

The notion of P_4-structure can be extended to F-structure for arbitrary fixed graphs F. Thus, it is a natural question to ask what is so special about the P_4 that makes it closely related to perfect graphs allowing results like the Semi Strong Perfect Graph Theorem ? As we will see in this section, there exists only one other graph, namely the P_3, that leads to results similar to the Semi Strong Perfect Graph Theorem.

For a fixed graph F, all graphs not containing F as an induced subgraph have the same F-structure. Thus any graph that is a candidate to replace the P_4 in the

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Semi Strong Perfect Graph Theorem must be an induced subgraph of all sufficiently long (i.e. of size at least $|F|$) odd holes and odd antiholes. This implies that the stability number and the clique number of F must have the value at most two.

Thus the only reasonable candidate for a statement similar to the Semi Strong Perfect Graph Theorem is the P_3 or its complement. However, it is easily seen that the perfection of a graph does *not* just depend on its P_3-structure. Figure 12 shows a well known counterexample.

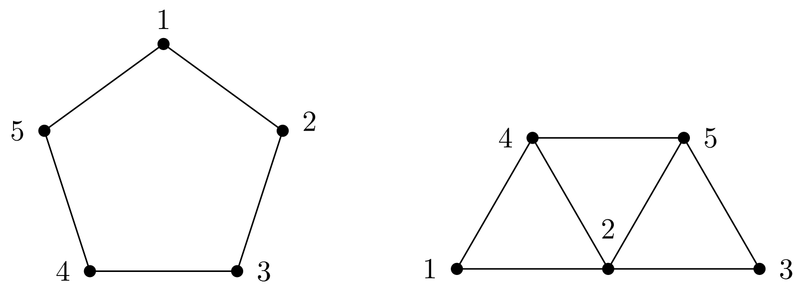

Figure 12: Two graphs with the same P_3-structure.

This simple counterexample leads astray as it seems to suggest that for larger graphs there exist much more counterexamples. However, one can show that this simple example is essentially the *only* exceptional case. More precisely we have

Theorem 12 (Hougardy 1996 [28])
A C_5-free graph is perfect if and only if it has the P_3-structure of a perfect graph. $\square$

In contrast to the P_4 the P_3 is not a self-complementary graph. Therefore Theorem 12 does not immediately imply the Perfect Graph Theorem. But using the Perfect Graph Theorem one easily gets the following stronger result.

Theorem 13 (Hougardy 1996 [28])
A C_5-free graph G is perfect if and only if G or $\overline{G}$ has the P_3-structure of a perfect graph. $\square$

Of course Theorem 13 implies the Perfect Graph Theorem, and the following result shows that the Strong Perfect Graph Conjecture implies Theorem 13. Therefore Theorem 12 can be seen as "another" Semi Strong Perfect Graph Theorem.

Theorem 14 (Hougardy 1996 [28])
Cycles of length at least six and complements of odd cycles of length at least seven have unique P_3-structure. $\square$

Even so Theorem 12 lacks the beauty of the Semi Strong Perfect Graph Theorem, it has in principle the same consequences as the Semi Strong Perfect Graph Theorem. A certificate to prove perfection for a given graph now would just consist of two parts. The first part is a perfect graph that has the same P_3-structure as the given graph. The second part is the proof that the given graph does not contain a C_5 as an induced subgraph. This is obviously easily done in polynomial time.

Since the P_3 is a subgraph of the P_4, one might think that Theorem 13 is already implied by the Semi Strong Perfect Graph Theorem. That this is not the case can be shown by the graphs in Figure 13a) and Figure 13b). This is a non-trivial example of two graphs that have the same P_3-structure but different P_4-structure. On the other hand there (of course) exist examples of graphs that have the same P_4-structure but different P_3- and $\overline{P}_3$-structure. One example is shown in Figure 13c) and Figure 13d).

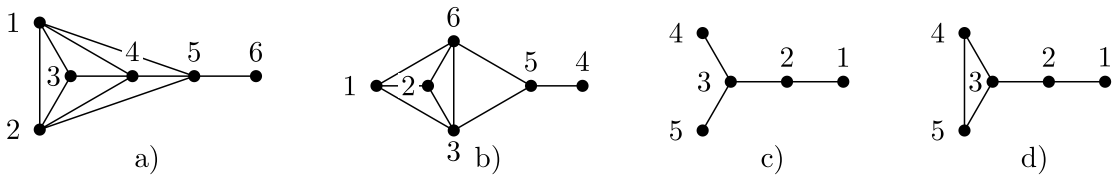

Figure 13: Two pairs of graphs showing that P_3-structure and P_4-structure are independent of each other.

Not very surprisingly, the asymptotic results that were stated in Section 5 for P_4-structure also hold for P_3-structure.

Theorem 15 (Hougardy 1996 [28])
Almost all (perfect) graphs have unique P_3-structure. □

The Problem P_4-StructureRecognition can be naturally extended from P_4 to any other graph F.

PROBLEM	F-StructureRecognition
Input:	A $\lvert F\rvert$-uniform hypergraph $\mathcal{H}$
Question:	Is there a graph having $\mathcal{H}$ as its F-structure ?

Currently, the P_3 is the only non-trivial graph different from the P_4 for which the complexity of this problem is known:

Theorem 16 (Hayward 1996 [25])

There exists a polynomial time algorithm to recognize the P_3-structures of graphs. □

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