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# Even pairs and the Strong Perfect Graph Conjecture

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**Abstract.** We will characterize all graphs that have the property that the graph and its complement are minimal even pair free. This characterization allows a new formulation of the Strong Perfect Graph Conjecture.

The reader is assumed to be familiar with perfect graphs (see for example [2]). A *hole* is a cycle of length at least five. An *odd hole* is a hole that has an odd number of vertices. An (odd) *anti-hole* is the complement of an (odd) hole. Berge's famous Strong Perfect Graph Conjecture may be formulated as follows [1]:

**Strong Perfect Graph Conjecture** *The only minimal imperfect graphs are the odd holes and the odd anti-holes.*

Two vertices in a graph are called an *even pair* if all induced paths between these two vertices have even length. Meyniel [6] has shown that minimal imperfect graphs are even pair free. A graph is called *minimal even pair free* if the graph does not contain an even pair but every induced subgraph contains an even pair or is a clique. We will show that proving that minimal imperfect graphs are *minimal* even pair free already implies the Strong Perfect Graph Conjecture.

**Proposition 1** *Odd holes are minimal even pair free.*

*Proof.* Obviously odd holes do not contain an even pair. Every proper induced subgraph of an odd hole is either disconnected or a path of length at least two or a clique. Thus every proper induced subgraph of an odd hole contains an even pair or is a clique. □

**Proposition 2** *Anti-holes are minimal even pair free.*

*Proof.* Let $G$ be an anti-hole. Then every two non-adjacent vertices of $G$ are connected by an induced path of length three and thus $G$ does not contain an even pair.

Let $H$ be a proper induced subgraph of $G$ that is not a clique. Then $\overline{H}$ contains a vertex of degree one and therefore $H$ contains a vertex $x$ that is connected to all but one vertex in $H$. All induced paths between $x$ and its non-neighbor have length two and thus $H$ contains an even pair. □

We are now able to give a characterization of all the graphs $G$ such that $G$ and $\overline{G}$ are minimal even pair free:

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**Lemma 1** *$G$ and $\overline{G}$ are minimal even pair free if and only if $G$ is an odd hole or an odd anti-hole.*

*Proof.* As shown in Proposition 1 and Proposition 2 odd holes and odd anti-holes are minimal even pair free.

To prove the other direction let $G$ be a graph such that $G$ and $\overline{G}$ are minimal even pair free. Suppose that $G$ is neither an odd hole nor an odd anti-hole. Then by Proposition 2 neither $G$ nor $\overline{G}$ contains an anti-hole as an induced subgraph. Therefore neither $G$ nor $\overline{G}$ can contain an induced cycle of length at least five. Graphs with this property are called *weakly triangulated* [3]. Hayward, Hoàng and Maffray [4] have shown that every weakly triangulated graph contains an even pair or is a clique. This contradicts the assumption that $G$ is minimal even pair free. $\square$

Using Lemma 1 one can easily derive a new equivalent formulation of the Strong Perfect Graph Conjecture in terms of minimal even pair free graphs:

**Lemma 2** *The following conjecture is equivalent to the Strong Perfect Graph Conjecture:*

*Minimal imperfect graphs are minimal even pair free.*

*Proof.* Let $G$ be a minimal imperfect graph. The Perfect Graph Theorem [5] implies that the complement of $G$ is again minimal imperfect. Thus if the above stated conjecture holds then by Lemma 1 $G$ is an odd hole or an odd anti-hole. This proves the Strong Perfect Graph Conjecture.

If the Strong Perfect Graph Conjecture holds then the only minimal imperfect graphs are the odd holes and the odd anti-holes thus Lemma 1 implies the above conjecture. $\square$

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