This paper appeared in: European Journal of Combinatorics 16 (1995), 17-21

Even and Odd Pairs in Linegraphs of Bipartite Graphs

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July 1993

Abstract. Two vertices in a graph are called an *even pair* (*odd pair*) if all induced paths between these two vertices have even (odd) length. Even and odd pairs have turned out to be of importance in conjunction with perfect graphs. We will characterize all linegraphs of bipartite graphs that contain an even resp. odd pair. In general it is a co-NP-complete problem to decide whether a graph contains an even pair. For the class of linegraphs of bipartite graphs we will show that testing for even resp. odd pairs can be done in polynomial time.

1 Introduction

Two vertices in a graph are called an *even pair* if every induced path between these two vertices has even length. A graph is called *strict quasi parity* if every induced subgraph contains an even pair or is a clique. The class of strict quasi parity graphs was introduced by Meyniel [11] and is denoted SQP for short.

A graph is called *minimal non strict quasi parity* if the graph does not belong to SQP but every proper induced subgraph does. Meyniel [11] has posed the problem to characterize all minimal non strict quasi parity graphs.

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To attack this problem we proposed after discussions with Chình Hoàng the following conjecture [8] which – if true – shows the significance of linegraphs of bipartite graphs in conjunction with even pairs:

Conjecture 1 *Every minimal non strict quasi parity graph is either*

i) an odd cycle of length at least five or

ii) the complement of a cycle of length at least seven or

iii) the linegraph of a bipartite graph.

For the class of planar graphs this conjecture has been proved recently by Linhares, Maffray, Preissmann and Reed [10].

The main motivation for studying strict quasi parity graphs is their connection to perfect graphs. A graph is called *perfect* if for every induced subgraph the chromatic number of the subgraph equals its clique number. A *minimal imperfect* graph is a graph that itself is not perfect but all its proper induced subgraphs are.

Meyniel [11] has shown that no minimal imperfect graph can contain an even pair and thereby proved that the graphs in SQP are perfect.

Many of the known classes of perfect graphs such as *bipartite graphs*, *comparability graphs*, *permutation graphs*, *interval graphs*, *triangulated graphs* – to mention just the 'classical' ones – are a subset of SQP. On the other hand there are several classes of perfect graphs like *strongly perfect graphs* [2], *BIP** [4], *alternately orientable graphs* [7] and *slim graphs* [6] that are supposed to be in SQP, but no one so far has been able to prove this. Here a better knowledge of minimal non strict quasi parity graphs would be very helpful.

Another link between perfect graphs and strict quasi parity graphs was established in [9]. Berge's famous Strong Perfect Graph Conjecture [1] states that a graph is perfect if and only if the graph and its complement do not contain odd induced cycles of length at least five. In [9] it was shown that this conjecture is equivalent to the statement that minimal imperfect graphs are minimal non strict quasi parity.

2 Even Pairs

If Conjecture 1 turns out to be true then the next problem in characterizing minimal non strict quasi parity graphs is to decide which linegraphs

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of bipartite graphs are minimal non strict quasi parity. The main result of this section procedes into this direction by giving a characterization of those linegraphs of bipartite graphs that contain an even pair.

Theorem 1 *The linegraph G of a bipartite graph H contains an even pair if and only if there exist two non-incident edges a_1a_2 and b_1b_2 in H such that a_1 and b_1 resp. a_2 and b_2 belong to the same colorclass of the bipartite graph H and to different components of $H - a_2 - b_2$ resp. $H - a_1 - b_1$.*

Proof. Let G be the linegraph of a bipartite graph H and let (a, b) be an even pair in G. By a_1a_2 resp. b_1b_2 we denote the edges in H which correspond to the vertices a resp. b in G. We may assume that a_1 and b_1 resp. a_2 and b_2 belong to the same colorclass of H. Since a and b cannot be adjacent the edges a_1a_2 and b_1b_2 are non-incident. Suppose now there exists a path in H between a_1 and b_1 that uses neither the vertex a_2 nor the vertex b_2. Then the length of this path must be even. A shortest such path will result in an odd induced path between a and b in G (see Figure 1).

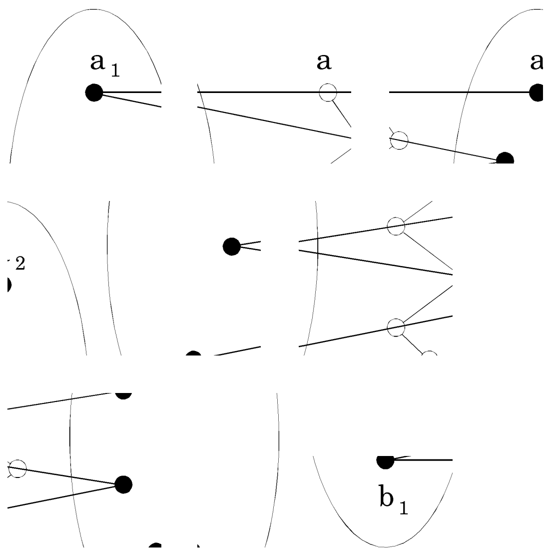

Figure 1

This contradicts the fact that (a, b) is an even pair in G. Thus we know that every path in H connecting a_1 and b_1 must contain at least one of the two vertices a_2 and b_2. This means that a_1 and b_1 belong to different components of $H - a_2 - b_2$. Using symmetrical arguments we see that a_2 and b_2 belong to different components of $H - a_1 - b_1$.

We now assume that H has the property stated in the theorem. We have to show that G contains an even pair. Let a and b be the vertices of G which correspond to the edges a_1a_2 and b_1b_2 in H. Since the edges a_1a_2 and b_1b_2 are non-incident the vertices a and b are non-adjacent. Suppose now that there exists an odd induced path between a and b in G. This path corresponds to an even path in H which connects the two edges a_1a_2 and b_1b_2. Taking a shortest such path and using symmetry we may assume that this path connects a_1 with b_1 and uses neither a_2 nor b_2. But such a path cannot exist since a_1 and b_1 belong to different components of $H - a_2 - b_2$. □

Remark Bienstock [3] has shown that deciding whether a graph contains an even pair is a co-NP-complete problem. Restricted to the class of linegraphs of bipartite graphs the above characterization shows that this problem can be solved in polynomial time. To this end one has to make use of the well known fact that a graph can be derived from its linegraph in polynomial time [14].

We can also give a characterization of complements of linegraphs of bipartite graphs that contain an even pair. It is well known that a linegraph of a bipartite graph cannot contain a diamond [5] (a diamond is the graph that is obtained from the complete graph on four vertices by removing one edge). Thus the desired characterization can easily be obtained from the following observation:

Lemma 1 *The complement of a diamond-free graph contains an even pair if and only if there exist two non-adjacent vertices which are not connected by an induced path of length three.*

Proof. The necessity of the above condition is obvious. The sufficiency follows from the simple fact that the complement of an odd path of length at least five contains a diamond. □

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Corollary 1 *The complement of a linegraph of a bipartite graph contains an even pair if and only if there exist two non-adjacent vertices which are not connected by an induced path of length three.*

Remark Again this characterization yields a polynomial time algorithm to detect even pairs in complements of linegraphs of bipartite graphs.

Obviously Theorem 1 together with Corollary 1 gives a characterization of linegraphs of bipartite graphs such that the graph and its complement are even pair free. Unfortunately these conditions are rather technical. We therefore want to give a simpler condition guaranteeing that a linegraph of a bipartite graph and its complement do not contain an even pair. To this end, we first prove the following lemma:

Lemma 2 *Let H be a bipartite graph of minimum degree at least three. Then the complement of its linegraph does not contain an even pair.*

Proof. Let G denote the linegraph of H. We have to prove that any two non-adjacent vertices a and b in $\overline{G}$ are connected by an odd induced path. Let a_1a_2 and b_1b_2 be the edges in H corresponding to the vertices a and b in $\overline{G}$. Since a and b are non-adjacent in $\overline{G}$ they are adjacent in G and therefore the edges a_1a_2 and b_1b_2 in H are incident. Thus we may assume that $a_1 = b_1$. Since H is bipartite the vertices a_2 and b_2 cannot be adjacent. Since by assumption these vertices have degree at least three there must exist two non-incident edges a_2v and b_2w such that $v \neq w$. (Take as v any neighbor of a different from a_1 and as w any neighbor of b_2 different from a_1 and v). Let x resp. y denote the vertices in G corresponding to the edges a_2v and b_2w in H. Then $ayxb$ is an induced P_4 in $\overline{G}$. □

Now we can easily give a sufficient condition for a linegraph of a bipartite graph such that the graph and its complement are even pair free.

Lemma 3 *Let H be a 3-connected bipartite graph. Then its linegraph and the complement of its linegraph do not contain an even pair.*

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Proof. Since H is 3-connected each vertex of H has degree at least three and therefore by Lemma 2 the complement of the linegraph of H is even pair free.

From Theorem 1 follows that also the linegraph of H cannot contain an even pair. □

Remark From Lemma 2 it also follows that the complement of a linegraph of a 3-edge-connected bipartite graph is even pair free. In contrast to this there exist arbitrarily high edge-connected bipartite graphs whose linegraphs contain an even pair. This is shown by the example in Figure 2.

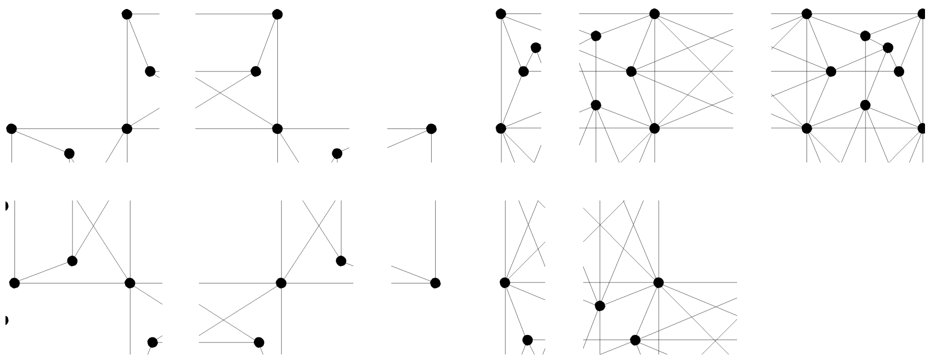

Figure 2: A bipartite graph and its linegraph.

The bipartite graph consists of four copies of the $K_{3,3}$ that are identified as shown in the figure. It is easy to see that this graph is 3-edge-connected. But two opposite vertices of the inner quadrilateral of its linegraph form an even pair.

By taking a $K_{p,p}$ instead of the $K_{3,3}$ one gets in an analogous way arbitrarily high edge connected bipartite graphs whose linegraphs contain an even pair.

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3 Odd Pairs

In this section we will show that nearly the same results as shown in the last section are valid for odd pairs. First we give a counterpart to Theorem 1 by giving a characterization of linegraphs of bipartite graphs that contain an odd pair.

Theorem 2 *The linegraph G of a bipartite graph H contains an odd pair if and only if there exist two non-incident edges a_1a_2 and b_1b_2 in H such that a_1 and b_1 resp. a_2 and b_2 belong to different colorclasses of the bipartite graph H and to different components of $H - a_2 - b_2$ resp. $H - a_1 - b_1$.*

Proof. The proof is similar to the proof of Theorem 1. Instead of looking at paths between a_1 and b_1 resp. a_2 and b_2 one has to look at paths between a_1 and b_2 resp. a_2 and b_1. Then the proof is essentially the same as for Theorem 1 and therefore omitted. □

We will now show that a counterpart of Lemma 3 also holds for odd pairs:

Lemma 4 *Let H be a 3-connected bipartite graph. Then its linegraph and the complement of its linegraph do not contain an odd pair.*

Proof. From Theorem 2 it follows that the linegraph of H cannot contain an odd pair.

We will now show that also $\overline{G}$, the complement of the linegraph of H, does not contain an odd pair. Let a and b be any two non-adjacent vertices of $\overline{G}$. These vertices are adjacent in G and therefore there correspond two incident edges of H to these two vertices. Since H is 3-connected and bipartite there must exist an edge in H that is not incident with any of these two edges. But this means that a and b are connected by a path of length 2 in $\overline{G}$. Therefore a and b do not form an odd pair. □

Remark As mentioned in the introduction Meyniel has shown that no minimal imperfect graph can contain an even pair. Whether an analogous statement holds for odd pairs is still an open question [12]. Nevertheless one could think of defining a class of graphs similar to the class SQP, but in terms of odd pairs. But since a diamond does not contain an odd pair and

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diamond-free Berge graphs are known to be perfect [13] this would not give an interesting new class of perfect graphs.

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