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Classes of Perfect Graphs

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Abstract. The Strong Perfect Graph Conjecture, suggested by Claude Berge in 1960, had a major impact on the development of graph theory over the last forty years. It has led to the definitions and study of many new classes of graphs for which the Strong Perfect Graph Conjecture has been verified. Powerful concepts and methods have been developed to prove the Strong Perfect Graph Conjecture for these special cases. In this paper we survey 120 of these classes, list their fundamental algorithmic properties and present all known relations between them.

1 Introduction

A graph is called *perfect* if the chromatic number and the clique number have the same value for each of its induced subgraphs. The notion of perfect graphs was introduced by Berge [6] in 1960. He also conjectured that a graph is perfect if and only if it contains, as an induced subgraph, neither an odd cycle of length at least five nor its complement.

This conjecture became known as the *Strong Perfect Graph Conjecture* and attempts to prove it contributed much to the developement of graph theory in the past forty years. The methods developed and the results proved have their uses also outside the area of perfect graphs. The theory of antiblocking polyhedra developed by Fulkerson [37], and the theory of modular decomposition (which has its origins in a paper of Gallai [39]) are two such examples.

The Strong Perfect Graph Conjecture has led to the definitions and study of many new classes of graphs for which the correctness of this conjecture has been verified. For several of these classes the Strong Perfect Graph Conjecture has been proved by showing that

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every graph in this class can be obtained from certain simple perfect graphs by repeated application of perfection preserving operations. By using this approach Chudnovsky, Robertson, Seymour and Thomas [19] were recently able to prove the Strong Perfect Graph Conjecture in its full generality. After remaining unsolved for more than forty years it can now be called the Strong Perfect Graph Theorem.

The aim of this paper is to survey 120 classes of perfect graphs. The criterion we used to include a class of perfect graphs in this survey is that its study be motivated by making progress towards a proof of the Strong Perfect Graph Conjecture. This criterion rules out including classes of perfect graphs that are known to be perfect just by definition, e.g. classes that are defined as subclasses of graphs already known to be perfect or classes that are defined as the union of two classes of perfect graphs. Some exceptions are made. For example we include some very basic classes such as trees or bipartite graphs. We have also included a few classes which were not known to contain only perfect graphs without using the Strong Perfect Graph Theorem. On the other hand, there probably exist several classes of perfect graphs which satisfy our criterion, but which are not included in this survey. We refer to [12, 13] for further information on graph classes.

A second motivation for studying perfect graphs besides the Strong Perfect Graph Conjecture are their nice algorithmic properties. While the problems of finding the clique number or the chromatic number of a graph are NP-hard in general, they can be solved in polynomial time for perfect graphs. This result is due to Grötschel, Lovász and Schrijver [47] from 1981. Unfortunately, their algorithms are based on the ellipsoid method and are therefore mostly of theoretical interest. It is still an open problem to find a combinatorial polynomial time algorithm to color perfect graphs or to compute the clique number of a perfect graph. However, for many classes of perfect graphs, such algorithms are known. In Section 4 we survey results of this kind. Moreover we consider the recognition complexity of all these classes, i.e. the question of deciding whether a given graph belongs to the class. Chudnovsky, Cornuejols, Liu, Seymour and Vušković [18] recently proved that there exists a polynomial time algorithm for recognizing perfect graphs. For several subclasses of perfect graphs such an algorithm is not yet known.

In many cases new classes of perfect graphs that have been introduced were motivated by generalizing known classes of perfect graphs. Many classes of perfect graphs are, therefore, subclasses of other classes of perfect graphs. We study the relation between all the classes of perfect graphs contained in this survey. The relations are given in the form of a table either stating that class A is contained in a class B or by giving an example of a graph showing that A is not a subclass of B. The table containing this information has 14400 entries. For several cases which had been open, the table answers the question whether a class A is a subclass of a class B.

The paper is organized as follows: Section 2 contains all basic notations used through-

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out this paper. The definitions of the classes of perfect graphs appearing in this paper are given in Section 3. In Section 4 we survey algorithms for the recognition and for solving optimization problems on classes of perfect graphs. The number of graphs contained in each of the classes of perfect graphs considered is given in Section 5. The relations between the classes of perfect graphs studied in this paper are presented in Section 6. All counterexamples that are needed to prove that certain classes are not contained in each other are described in Section 7.

2 Notation

Given a graph $G = (V, E)$ with vertex set V and edge set E we denote by n and m the cardinality of V and E. The *degree* of a vertex is the number of edges incident to this vertex. The *maximum degree* $\Delta(G)$ is the largest degree of a vertex of G. A *k-coloring* of the vertices of a graph $G = (V, E)$ is a map $f : V \to \{1, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever $\{x, y\}$ is an edge in G. The *chromatic number* $\chi(G)$ is the least number k such that G admits a k-coloring. A *clique* is a graph containing all possible edges. A clique on i vertices is denoted by K_i. The *clique number* $\omega(G)$ of a graph G is the size of a largest clique contained in G as a subgraph. A *stable set* in a graph is a set of vertices no two of which are adjacent. By I_i we denote a stable set of size i. The *stability number* $\alpha(G)$ is the size of a largest stable set in G. The *complement* $\overline{G}$ of a graph G has the same vertex set as G and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. Obviously, we have $\alpha(G) = \omega(\overline{G})$, and the *clique covering number* $\theta(G)$ is defined as $\chi(\overline{G})$.

A graph is called *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H. A *hole* is a chordless cycle of length at least four and an *antihole* is the complement of a hole. An *odd* (respectively *even*) hole is a hole with an odd (respectively even) number of vertices. A graph is called *Berge* if it contains no odd holes and no odd antiholes as induced subgraphs. A *star-cutset* in a graph G is a subset C of vertices such that $G \setminus C$ is disconnected and such that some vertex in C is adjacent to all other vertices in C.

A complete bipartite graph, i.e. a bipartite graph with all possible edges between the vertices of the two color classes of size r and s, respectively, is denoted by $K_{r,s}$. A $K_{1,3}$ is called a *claw*. A path on i vertices is denoted by P_i and a cycle on i vertices by C_i. The two vertices of degree one in a path are called the *endpoints* of the path. In a P_4 the vertices of degree two are called *midpoints* of the P_4. The two edges of a P_4 incident to the endpoints of the P_4 are called *wings*. The *wing graph* $W(G)$ of a graph G has as its vertices all edges of G and two edges are adjacent in $W(G)$ if there is an induced P_4 in G that has these two edges as its wings. Given a graph G its *k-overlap graph* is

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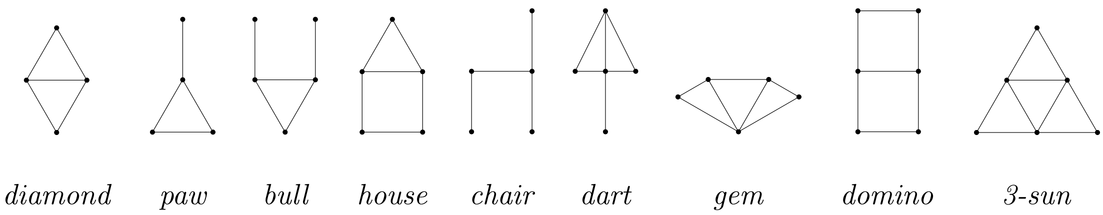

Figure 1: Some small graphs with special names.

defined as the graph whose vertices are all induced P_4's of G and in which two vertices are adjacent if the corresponding P_4's in G have exactly k vertices in common. Two vertices x, y in a graph are called *partners* if there exist vertices u, v, w distinct from x, y such that $\{x, u, v, w\}$ and $\{y, u, v, w\}$ each induce a P_4 in the graph. The *partner graph* of a graph G is the graph whose vertices are the vertices of G and whose edges join pairs of partners in G.

Two vertices form an *even pair* if all induced paths between these two vertices have even length. The *line graph* $L(G)$ of a graph G is the graph that has the edges of G as vertices and in which two vertices in $L(G)$ are adjacent if the corresponding edges of G are adjacent (that is, share a vertex). Some small graphs are given special names. Figure 1 contains such graphs with the names that are used throughout this paper.

3 Definitions of Graph Classes

In this section we briefly present in alphabetical order the definitions of all classes of perfect graphs appearing in this paper. For each class we give a reference to a proof that all graphs in the class are perfect. Note that with the proof of the Strong Perfect Graph Conjecture it follows immediately for all classes that they contain only perfect graphs.

alternately colorable A graph is called alternately colorable if its edges can be colored using only two colors in such a way that in every induced cycle of length at least four no two adjacent edges have the same color. This class of graphs has been defined by Hoàng [61] who also proved the perfectness of these graphs.

alternately orientable A graph is called alternately orientable if it admits an orientation of its edges such that in every induced cycle of length at least four the orientation of the edges alternates. This class of graphs was defined by Hoàng [61] who also proved the perfectness of these graphs.

AT-free Berge A graph is called AT-free Berge if it is a Berge graph and does not

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contain an asteroidal triple. An asteroidal triple is an independent set of three vertices such that each pair is joined by a path that avoids the neighborhood of the third. This class of graphs was introduced in [80]. Perfectness of these graphs was observed by Maffray [29, page 401]. As his argument is unpublished we briefly state it here. If an AT-free Berge graph has stability number two then it must be the complement of a bipartite graph and therefore perfect. If the graph has a stable set of size three, say $\{x, y, z\}$, then since the graph is AT-free it must be that the set of all neighbours of one of them, say z, separates x from y, i.e., z is the center of a star-cutset. Perfection follows from [21].

BIP* A graph belongs to the class BIP* if all induced subgraphs H which are not bipartite have the property that H or $\overline{H}$ contains a star–cutset. This class of graphs was defined by Chvátal [21] who also proved the perfectness of these graphs.

bipartite A graph is called bipartite if its chromatic number is at most two. Perfectness of bipartite graphs follows from the definition.

brittle A graph is called brittle if every induced subgraph H of G contains a vertex that is not an endpoint or not a midpoint of a P_4 in H. This class of graphs was introduced by Chvátal. Perfection follows easily as all brittle graphs are perfectly orderable [63].

bull-free Berge A bull-free Berge graph is a Berge graph that does not contain a bull (see Figure 1) as an induced subgraph. Chvátal and Sbihi [24] proved that these graphs are perfect.

C_4-free Berge A C_4-free Berge graph is a Berge graph that does not contain a cycle on four vertices as an induced subgraph. Perfection of these graphs was shown by Conforti, Cornuéjols, and Vušković [28].

chair-free Berge A chair-free Berge graph is a Berge graph that does not contain a chair (see Figure 1) as an induced subgraph. Perfection of these graphs was shown by Sassano [107].

chordal see →*triangulated*.

claw-free Berge A graph is claw-free Berge if it is a Berge graph that does not contain a $K_{1,3}$ (which is called a claw) as an induced subgraph. Parthasarathy and Ravindra [96] proved the perfectness of these graphs.

clique-separable A graph is called clique-separable if every induced subgraph that does not contain a clique-cutset is of one of the following two types. Either it is a complete multipartite graph or its vertex set can be partitioned into two sets V_1

and V_2 such that V_1 is a connected bipartite graph, V_2 is a clique and all vertices in V_1 are connected to all vertices in V_2. This class of graphs appears first in the paper of Gallai [38]. Gavril [41] invented the name for this class. Perfection follows immediately from the definition.

co-*class* Complements of the graphs in →*class*.

cograph see →P_4*-free*.

cograph contraction A graph G is a cograph contraction if there exists a cograph H and some pairwise disjoint independent sets in H such that G is obtained from H by contracting each of the independent sets to a single vertex (resulting multiple edges are identified) and joining the new vertices pairwise. Hujter and Tuza [73] introduced this class of graphs and proved that they are perfect. A good characterization of these graphs is given in [79].

comparability A graph is a comparability graph if there exists a partial order "$<$" on its vertices such that two vertices x and y are adjacent in the graph if and only if $x < y$ or $y < x$. These graphs are also called *transitively orientable*. Perfectness follows from a classical result of Dilworth [33].

$\Delta \leq 6$ **Berge** The class $\Delta \leq 6$ Berge contains all Berge graphs in which the maximum degree is at most 6. Grinstead [46] proved that these graphs are perfect.

dart-free Berge A graph is dart-free Berge if it is a Berge graph that does not contain a dart (see Figure 1) as an induced subgraph. Sun [114] proved the perfectness of these graphs.

degenerate Berge A graph is called degenerate Berge if it is a Berge graph and every induced subgraph H has a vertex of degree at most $\omega(H) + 1$. This class of graphs has been defined by Aït Haddadène and Maffray [1] who also proved the perfectness of these graphs.

diamond-free Berge A graph is diamond-free Berge if it is a Berge graph that does not contain a diamond (a K_4 with one edge removed, see Figure 1) as an induced subgraph. Tucker [119] proved the perfectness of these graphs based on earlier results of Parthasarathy and Ravindra [97].

doc-free Berge The name doc-free Berge is an abbreviation for the class of diamonded odd cycle-free Berge graphs. These are Berge graphs that do not contain diamonded odd cycles as induced subgraphs. A diamonded odd cycle on five vertices is a P_4 or a C_4 together with a fifth vertex joined to all the others. An odd cycle C with more than five vertices is called a diamonded odd cycle if it has two chords $\{x, y\}$

and $\{x, z\}$ with $\{y, z\}$ an edge of C and there exists a vertex w not on C adjacent to y and z but not x. Moreover no edge of C other than $\{y, z\}$ is on a triangle induced by the vertices of C. Carducci [17] proved the perfectness of doc-free Berge graphs.

elementary A graph is called elementary if its edges can be colored by two colors so that no monochromatic induced P_3 occurs. Equivalently these are graphs whose Gallai-graph is bipartite. Elementary graphs were introduced by Chvátal and Sbihi [25]. Perfectness of these graphs follows from the fact that they are claw-free Berge. Maffray and Reed [84] give a description of the structure of elementary graphs.

forest A graph is called a forest if it does not contain a cycle. These graphs are perfect as they are bipartite.

Gallai There exist two different classes of perfect graphs which have been given the name Gallai. Historically → *triangulated graphs* were called Gallai graphs [9]. Later, → *i-triangulated graphs* were given this name.

gem-free Berge A graph is called gem-free Berge if it is a Berge graph without a gem (see Figure 1) as an induced subgraph. Perfection of these graphs follows from the Strong Perfect Graph Theorem [19].

HHD-free A graph is called HHD-free if it does not contain a house (see Figure 1), a hole of length at least 5 or a domino (see Figure 1) as an induced subgraph. This class of graphs was introduced in [63]. Perfectness follows easily from the observation that these graphs are Meyniel.

Hoàng A graph is called Hoàng if its wing graph (see Section 2) is bipartite. This class of graphs was introduced in [22]. Perfection of these graphs follows from the Strong Perfect Graph Theorem [19].

***i*-triangulated** A graph is called i-triangulated if every odd cycle of length at least five has two non-crossing chords. These graphs are also called →*Gallai*. Gallai [38] proved the perfectness of these graphs.

I_4-free Berge A graph is I_4-free Berge if it is Berge and does not contain a stable set on four vertices. These are complements of → K_4-free Berge graphs.

interval A graph is an interval graph if each vertex can be represented by an interval on the real line in such a way that two vertices are adjacent if and only if their corresponding intervals intersect. These graphs are → *triangulated* [43] and therefore perfect.

K_4-free Berge A graph is K_4-free Berge if it is Berge and does not contain a clique on four vertices. Tucker [118] proved the perfectness of these graphs.

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(K_5, P_5)-free Berge A graph is (K_5, P_5)-free Berge if it is Berge and does not contain a K_5 or a P_5 as an induced subgraph. Perfectness of these graphs was proved by Maffray and Preissmann [82].

LGBIP The class LGBIP consists of all line graphs (see Section 2) of bipartite graphs. As noted in [6] perfection of these graphs follows from a classical result of König [78].

line perfect A graph is called line perfect if its line graph is perfect. Perfection of these graphs follows from a characterization of Trotter [116].

locally perfect A graph is called locally perfect if every induced subgraph admits a coloring of its vertices such that for any vertex the number of colors used in the neighborhood of this vertex equals the clique number of the neighborhood of the vertex. This class of graphs was introduced by Preissmann [98] who also proved the perfection of these graphs.

Meyniel A graph is called Meyniel if every odd cycle of length at least five has at least two chords. Meyniel [87, 88] proved the perfectness of these graphs. The same result was proven independently by Markosian and Karapetian [86].

murky A graph is called murky if it contains no C_5, P_6 or $\overline{P}_6$ as an induced subgraph. Hayward [52] proved that murky graphs are perfect.

1-overlap bipartite A graph belongs to the class 1-overlap bipartite if it is C_5-free and its 1-overlap graph (see Section 2) is bipartite. Hoàng, Hougardy and Maffray [62] proved that these graphs are perfect.

opposition A graph is called opposition if it admits an orientation of its edges such that in every induced P_4 the two end edges both either point inwards or outwards. This class of graphs was introduced by Chvátal [22]. Perfection follows from the Strong Perfect Graph Theorem [19]. Note that there is another class of perfect graphs called opposition [92] which additionally requires that the orientation of the edges be acyclic. Therefore we call this class → *strict opposition*.

P_4-free A graph is called P_4-free if it does not contain a P_4 as an induced subgraph. These graphs are also called cographs. Perfection follows from a result of Seinsche [110].

P_4-lite A graph is called P_4-lite if every induced subgraph H with at most six vertices contains either at most two induced P_4's or H or $\overline{H}$ is the 3-sun (see Section 2). These graphs were introduced in [76]. Perfection follows from the fact that they are → *weakly triangulated*.

P_4-reducible A graph is called P_4-reducible if every vertex belongs to at most one

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induced P_4. These graphs were introduced in [75]. Perfection follows from the fact that they are → *weakly triangulated.*

P_4-sparse A graph is called P_4-sparse if no set of five vertices induces more than one P_4. This class of graphs was introduced in [60]. Perfection follows from the fact that these graphs are → *weakly triangulated.*

P_4-stable Berge A graph is called P_4-stable Berge if it is a Berge graph containing a stable set that intersects all induced P_4's. Hoàng and Le [64] proved that these graphs are perfect.

parity A graph is called parity if for every pair of nodes, the lengths of all induced paths connecting them have the same parity. Burlet and Uhry [16] proved that a graph is parity if and only if each odd cycle of length at least five has two crossing chords. Perfection of these graphs was proved by Olaru [94].

partner-graph triangle-free The class partner-graph triangle-free contains all graphs whose partner graph (see Section 2) is triangle free. Perfection of this class of graphs was proved by Hayward and Lenhart [54].

paw-free Berge A graph is called paw-free Berge if it is a Berge graph that does not contain a paw (see Figure 1) as an induced subgraph. Perfection follows from the observation that these graphs are Meyniel. See [93] for a characterization of paw-free graphs.

perfectly contractile A graph is called perfectly contractile if for any induced subgraph H there exists a sequence $H = H_0, H_1, \ldots, H_k$ for some k such that H_{i+1} is obtained from H_i by contraction of an even pair (see Section 2) and H_k is a clique. Bertschi [10] introduced this class of graphs and proved that they are perfect.

perfectly orderable A graph is called perfectly orderable if there exists an acyclic orientation of the edges such that in no induced P_4 the two end edges are oriented inwards. This class of graphs was introduced by Chvátal [20] who also proved that they are perfect.

permutation A graph is called a permutation graph if it can be represented by a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ in such a way that two vertices $i < j$ are adjacent if and only if $\pi(i) > \pi(j)$. Perfection of these graphs follows from a characterization of Dushnik and Miller [35].

planar Berge The class planar Berge contains all Berge graphs that are planar. Perfection of these graphs was shown by Tucker [117].

preperfect A vertex x in a graph G is called predominant if there exists another vertex

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y such that every maximum clique of G containing y contains x or every maximum stable set containing x contains y. A graph is called preperfect if every induced subgraph has a predominant vertex. Hammer and Maffray [49] introduced this class of graphs and proved that all preperfect graphs are perfect.

quasi-parity A graph is called quasi-parity if for every induced subgraph H of G either H or $\overline{H}$ contains an even pair (see Section 2). Meyniel [89] proved that quasi-parity graphs are perfect.

Raspail A graph is called Raspail if every odd cycle has a short chord, i.e. a chord joining two vertices that have distance two on the cycle. See [114] for an explanation of where the name for this class comes from. Perfection of these graphs follows from the Strong Perfect Graph Theorem [19].

skeletal A graph is called skeletal if it can be obtained by removing a collection $\mathcal{S}$ of stars in a $\rightarrow$ *parity* graph. No two centers of stars in $\mathcal{S}$ must be joined by an induced path of length at most two. Hertz [58] proved that these graphs are perfect.

slender A graph is called slender if it can be obtained from an $\rightarrow$*i-triangulated* graph by deleting all the edges of an arbitrary matching. Hertz [57] proved that these graphs are perfect.

slightly triangulated A graph is called slightly triangulated if it contains no hole of length at least five and every induced subgraph H contains a vertex whose neighborhood in H does not contain a P_4. This class of graphs was introduced by Maire [85] who also proved the perfectness of these graphs.

slim A graph is called slim if it can be obtained from a Meyniel graph by removing all the edges that are induced by an arbitrary vertex set. Hertz [56] proved that slim graphs are perfect.

snap A graph is called snap if it is Berge and every induced subgraph contains a vertex whose neighborhood can be partitioned into a stable set and a clique. Maffray and Preissmann [83] proved the perfection of snap graphs.

split A graph is called split if its vertex set can be partitioned into two sets V_1 and V_2 such that V_1 induces a stable set and V_2 induces a clique. Perfection of split graphs follows from the fact that they are triangulated.

strict opposition A graph is called strict opposition if it admits an acyclic orientation of its edges such that in every induced P_4 the two end edges both either point inwards or outwards. Olariu [92] proved that these graphs are perfect.

strict quasi-parity A graph is called strict quasi-parity if every induced subgraph

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either contains an even pair (see Section 2) or is a clique. Meyniel [89] proved that strict quasi-parity graphs are perfect.

strongly perfect A graph is called strongly perfect if every induced subgraph contains a stable set that intersects all maximal cliques. Berge and Duchet [8] introduced strongly perfect graphs and proved their perfection.

3-overlap bipartite A graph belongs to the class 3-overlap bipartite if its 3-overlap graph (see Section 2) is bipartite. Hoàng, Hougardy and Maffray [62] proved that these graphs are perfect.

3-overlap triangle free A graph belongs to the class 3-overlap bipartite if it is Berge and its 3-overlap graph (see Section 2) is triangle free. Hoàng, Hougardy and Maffray [62] proved that these graphs are perfect.

threshold A graph is called a threshold graph if it does not contain a C_4, $\overline{C}_4$ and P_4 as an induced subgraph. Perfection of these graphs follows easily as they are triangulated.

totally unimodular see →*unimodular*.

transitively orientable see →*comparability*.

tree A connected graph that does not contain a cycle is called a tree. Trees are perfect as they are bipartite.

triangulated A graph is called triangulated if every cycle of length at least four contains a chord. These graphs are also called *chordal*. Perfection of triangulated graphs follows from results of Hajnal and Surányi [48] and Dirac [34].

trivially perfect A graph is called trivially perfect if for each induced subgraph H the stability number of H equals the number of maximal cliques in H. Golumbic [44] introduced these graphs and proved their perfection. He also showed that a graph is trivially perfect if and only if it contains no C_4 and no P_4 as an induced subgraph.

2-overlap bipartite A graph belongs to the class 2-overlap bipartite if it is C_5-free and its 2-overlap graph (see Section 2) is bipartite. Hoàng, Hougardy and Maffray [62] proved that these graphs are perfect.

2-overlap triangle free A graph belongs to the class 2-overlap triangle-free if it is Berge and its 2-overlap graph (see Section 2) is triangle free. Hoàng, Hougardy and Maffray [62] proved that these graphs are perfect.

2-split Berge A graph is called 2-split Berge if it is a Berge graph and if it can be