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Recursive generation of partitionable graphs*

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Abstract

Results of Lovász (1972) and Padberg (1974) imply that partitionable graphs contain all the potential counterexamples to Berge's famous Strong Perfect Graph Conjecture. A recursive method of generating partitionable graphs was suggested by Chvátal, Graham, Perold and Whitesides (1979). Results of Sebő (1996) entail that Berge's conjecture holds for all the partitionable graphs obtained by this method. Here we suggest a more general recursion. Computer experiments show that it generates *all* the partitionable graphs with $\omega = 3, \alpha \leq 9$ (and we conjecture that the same will hold for bigger α, too) and many but not all for $(\omega, \alpha) = (4, 4)$ and $(4, 5)$. Here α and ω are respectively the clique and stability numbers of a partitionable graph, that is the numbers of vertices in its maximum cliques and stable sets. All the partitionable graphs generated by our method contain a *critical ω-clique*, that is an ω-clique which intersects only $2\omega - 2$ other ω-cliques. This property might imply that in our class there are no counterexamples to Berge's conjecture (cf. Sebő (1996)), however this question is still open.

1 Introduction

Given a graph G, we denote by $n = n(G)$ the number of vertices in G, by $\omega = \omega(G)$ its *clique number*, that is the maximal number of pairwise connected vertices, by $\alpha = \alpha(G)$ its *stability number*, that is the maximal number of pairwise non-connected vertices, and by $\chi = \chi(G)$ its *chromatic number*, that is the minimal number of color classes (i.e. stable sets) covering all vertices of G.

In (1961) Claude Berge introduced the notion of a perfect graph. A graph G is called *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' in G. Naturally, a graph G is called *minimally imperfect* if it is a vertex-minimal non-perfect graph, or in other words if G itself is not perfect but every proper induced subgraph G' of G is perfect. It is not difficult to see that chordless odd cycles of length five or more (*odd holes*) as well as their complements (*odd anti-holes*) are minimally imperfect. Berge conjectured that there are no other minimally imperfect graphs. This conjecture is called the *Strong Perfect Graph Conjecture* and it is still open. A weaker conjecture, stating that the complement G^c of a perfect graph G is perfect was also suggested by Berge (1961) and was proved by Lovász (1972) (it is known as the *Perfect Graph Theorem*.)

We would like to recall here two important results from the paper by Lovász (1972). The first one is stating that a graph G is perfect if and only if $n(G') \leq \alpha(G')\omega(G')$ for every induced subgraph G' in G. Since the equalities $\alpha(G) = \omega(G^c)$ and $\omega(G) = \alpha(G^c)$ obviously hold for every graph G, the above inequality implies readily the Perfect Graph Theorem.

The second one states that every minimally imperfect graph G is *partitionable*, that is $n(G) = \alpha(G)\omega(G)+1$, and for every vertex v the induced subgraph $G(V \setminus \{v\})$ can be partitioned into $\alpha(G)$ cliques of size $\omega(G)$, as well as into $\omega(G)$ stable sets of size $\alpha(G)$. If G is partitionable then clearly $\chi(G) = \omega(G) + 1$, $\chi(G(V \setminus \{v\})) = \omega(G) = \omega(G(V \setminus \{v\}))$, and thus the complementary graph G^c is partitionable, too.

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Padberg (1974) derived from Lovász' result that for any minimally imperfect graph G the number of ω-cliques is n and every vertex belongs to exactly ω of the ω-cliques. The characteristic vectors of these maximum cliques form a basis in $\mathbb{R}^n$. Padberg also observed the following convenient way to list all n maximum cliques (of size ω) of G. Let us fix an arbitrary ω-clique C and for every vertex $v \in C$ consider a partition of $G(V \setminus \{v\})$ into α maximum cliques. Such a partition is unique. There are ω different vertices $v \in C$ and there are α maximum cliques in each partition. All these cliques appear to be different. Together with the clique C itself we get exactly $\alpha\omega + 1 = n$ maximum cliques of G. Of course, the analogous construction and relations take place for maximum stable sets, too.

Bland, Huang and Trotter (1979) proved that all these properties hold not only for minimally imperfect but for arbitrary partitionable graphs as well.

Due to Padberg's construction, it is immediate to see that in every partitionable graph G every ω-clique C intersects at least $2\omega - 2$ other ω-cliques of G. (This fact was proved for minimally imperfect graphs by Olaru (1973); see also Reed (1986) and Sebő (1996a)). Indeed, let us choose another ω-clique C', disjoint from C, and consider the clique partitions corresponding to the vertices of C'. Every ω-clique of G (except C') appears in these partitions exactly once, hence exactly one of these partitions contains C. Thus, every other partition splits C in at least two parts, and therefore C must meet at least $2\omega - 2$ other ω-cliques of G.

An ω-clique is called *critical* if it intersects *exactly* $2\omega - 2$ other ω-cliques. An edge $e \in E(G)$ of a partitionable graph G is called *critical* if $\alpha(G - e) = \alpha(G) + 1$, or in other words, if there exist two maximum stable sets S and S' which have $\alpha(G) - 1$ vertices in common and the two vertices in their symmetric difference are connected by the edge e.

Critical cliques and critical edges are in fact strongly related. This relation was studied by Sebő (1996b). He proved (see Lemma 3.1 of Sebő (1996b)) that for every critical ω-clique C of an (α, ω)-partitionable graph G, the following claims are equivalent:

(i) C is a critical clique;

(ii) The critical edges in C form a spanning tree T_C of C;

(iii) The induced subgraph $G(V \setminus C)$ is uniquely colorable,

where a graph H is called *uniquely colorable* if it has a unique partition into $\chi(H)$ stable sets.

It is immediate from the above observations that the $2\omega - 2$ cliques intersecting a critical clique C can be combined into $\omega - 1$ pairs such that each of these pairs induces a partition of the vertices of C into two nonempty parts. Furthermore, as observed in the proof of the above cited result by Sebő (1996b), these partitions correspond to the critical edges in C. Namely, the removal of any edge $e \in E(T_C)$ splits T_C into two connected components, hence splitting the vertices of C into two parts. The $2\omega - 2$ sets obtained in this way, corresponding to the $\omega - 1$ edges of T_C, are exactly the $2\omega - 2$ non-empty intersections of C with the other ω-cliques of G.

These observations suggest the following reduction, the correctness of which we shall show in Section 5. Given a partitionable (α, ω)-graph G which contains a critical clique C, let us consider the tree $T = T_C$ formed by the critical edges in C. Let us now consider any pair of disjoint ω-cliques C' and C'', corresponding to an edge e of T, that is for which the intersections $C \cap C'$ and $C \cap C''$ are nonempty and form the same partition of C as the one obtained by the removal of the edge e. Let us now change the graph by changing the list of its maximum cliques in the following way. Remove the cliques C', C'' and instead of these two add only one new ω-clique $(C' \setminus C) \cup (C'' \setminus C)$. Let us repeat the same for all the $\omega - 1$ pairs of ω-cliques, corresponding to the edges of T. Finally, let us remove the clique C itself from the list. We shall prove in Section 5 that this procedure *always* results in a new *partitionable* $(\alpha - 1, \omega)$-graph G'.

The main result of this paper is to describe an inverse, recursive, procedure which constructs a new partitionable (α, ω)-graph G from a given partitionable $(\alpha - 1, \omega)$-graph G'. In this procedure we increment the vertex set of G' by ω new vertices, and specify a spanning tree T on the set of new vertices. If we can find and split $\omega - 1$ maximum cliques of G' satisfying certain combinatorial properties, related to the

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structure of T, then the recursive procedure can be carried out yielding a new partitionable (α, ω)-graph G. In Section 6 we describe necessary and sufficient conditions for this procedure to work, while in Section 7 we demonstrate that this procedure can indeed be carried out if G' is a "web", and T is chosen as a "spider", or in other words, we prove that it is always possible "to insert a spider into a web".

Let us add that the newly added ω vertices will always form a critical clique in the resulting partitionable (α, ω)-graph G, and every partitionable (α, ω)-graph which has a critical clique can be obtained in this way. It is natural to ask how many partitionable graphs have critical cliques. We conjecture that in case $\omega = 3$ they all have. Computations confirm this conjecture for $\alpha < 10$, see Section 8. It is easy to see that this conjecture is equivalent to the following one. Every partitionable $(\alpha, 3)$-graph contains an induced *gem*, that is a graph on five vertices $\{a, b, c, d, e\}$ having the pairs $\{(a, b), (b, c), (c, d), (d, e), (a, c), (c, e), (b, d)\}$ as edges (cf. Lemma 8 in Section 8). However, it is not even known if every $(\alpha, 3)$-graph contains an induced *diamond*, that is a graph on four vertices $\{a, b, c, d\}$ having the pairs $\{(a, b), (b, c), (c, d), (a, c), (b, d)\}$ as edges.

Computer enumeration shows that for $\omega = 4$ there are partitionable graphs without critical cliques, see Section 8. Namely, there exist 5 non-isomorphic partitionable $(3, 4)$-graphs and all 5 have critical cliques, there are 132 non-isomorphic partitionable $(4, 4)$-graphs out of which 126 have critical cliques and 6 do not, and there are 8340 non-isomorphic partitionable $(5, 4)$-graphs out of which only 6909 have critical cliques.

Even though not all partitionable graphs contain a critical clique, minimally imperfect graphs are conjectured to have all of their cliques to be critical. More precisely, Sebő (1992) conjectured that for every minimally imperfect graph G and every maximum clique C of G the graph $G \setminus C$ is uniquely colorable, and showed that this conjecture is in fact equivalent with the Strong Perfect Graph Conjecture. According to the above cited equivalences this is also equivalent with the conjecture that all maximum cliques of a minimally imperfect graph are critical. It was also shown by Sebő (1996a) that a partitionable graph G cannot be a counterexample to Berge's conjecture if both G and its complement G^c contain critical cliques.

Let us remark next that our recursion generalizes an analogous one suggested by Chvátal, Graham, Perold and Whitesides (1979). We get their recursion as a special case when the tree $T = T_C$ on the new vertices C is chosen as a simple path and the $\omega - 1$ maximum cliques in G, which define the recursion, form a chain on $2\omega - 2$ vertices, that is satisfy that $C_k = \{v_k, v_{k+1}, ..., v_{k+\omega-1}\}$, for $k = 1, ..., \omega - 1$.

Let us note that the resulting partitionable graph depends not only on the structure of the tree $T = T_C$ spanning the new vertices, but also on the choice of the $\omega - 1$ maximum cliques used in the recursion. For example, if $\omega = 3$ then there exists only one spanning tree with 2 edges (a simple path P_3), but still we can choose two 3-cliques C_1 and C_2 in several different ways, e.g. such that the cardinality of the intersection $|C_1 \cap C_2|$ is 2,1 or 0. Chvátal, Graham, Perold and Whitesides (1979) demonstrated that using $|C_1 \cap C_2| = 2$, only 4 out of 5 partitionable (4,3)-graphs can be recursively generated. Our computation shows that the fifth one can be generated by using the recursion of Section 6 and choosing cliques with $|C_1 \cap C_2| = 1$. Moreover, to obtain all $(7, 3)$-graphs, all intersection sizes $0 \leq |C_1 \cap C_2| \leq 2$ are necessary to consider. Let us finally note that a trivial application of this recursion produces a $(2k + 1)$-hole from a $(2k - 1)$-hole, and if applied to the complementary graph in a natural way, then produces a $(2k + 1)$-antihole from a $(2k - 1)$-antihole. Hence, starting with C_5 all odd holes and antiholes can indeed be obtained by this procedure, trivially.

Our method is based on purely combinatorial, elementary proofs. To demonstrate the consistency of this approach as well as for the sake of completeness, we include a short proof for every statement we use.

In Section 2 we introduce the notion of a *partitionable* hypergraph, and then prove that such a hypergraph $\mathcal{C}$ determines essentially uniquely a partitionable graph G in which the edges of $\mathcal{C}$ are the maximum cliques. This result is essential for the proofs of correctness of both the reduction and the recursion procedures in Sections 5 and 6, since it makes possible to define a partitionable graph by specifying only the family of its maximum cliques.

Motivated by the intersection structure of the maximum cliques of a partitionable graph, in Section 3 we consider special hypergraphs, so called *tree-coverings*, and show that such a hypergraph on ω vertices must always have at least $2\omega - 2$ hyperedges. Moreover, we show that a tree-covering of exactly $2\omega - 2$ hyperedges uniquely determines a spanning tree on the ω vertices. When a tree-covering arises as the intersections of the maximum cliques in a partitionable (α, ω)-graph G with a critical clique C, we prove that the edges of the uniquely corresponding spanning tree T on C are all critical; we obtain a highly regular decomposition

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of G, see Section 4; and we also show that the number of leaves (and hence all the degrees of the vertices) in such a tree T cannot exceed α.

Let us add finally that investigating the structure of the spanning trees arising in this way from critical cliques may be an interesting problem on its own. The webs give an example of partitionable graphs in which critical edges in a critical clique form a simple path, and the construction of Section 7 demonstrates that "spiders" can also arise in this way. We are, however, not aware of any other examples.

2 Axioms for partitionability

In their definition of partitionable graphs Bland, Huang and Trotter (1979) demand partitionability for both families of maximum cliques $\mathcal{C}$ and maximum stable sets $\mathcal{S}$. But in fact, it is sufficient to demand partitionability for only one of these two families which then will imply the partitionability of the other one. This idea is not new, and some results in this direction can be found in the literature. For completeness, we devote a special section to this problem, as well as to some other simple axioms which characterize partitionable graphs. This section plays an important role in our paper, because the reduction and recursion, which we will introduce, are based on transformations of the family of ω-cliques, only. The justification of this approach is based on the following subsection.

2.1 A one-axiom definition

Let us consider a finite set V of n elements, and a family $\mathcal{C}$ of its subsets.

Definition 1 *The family $\mathcal{C}$ will be called* partitionable *if $|\mathcal{C}| \leq |V|$ and for every $v \in V$ the set $V \setminus \{v\}$ is a union of some pairwise disjoint sets from $\mathcal{C}$, in other words, if for every $v \in V$ there exists a subfamily $\mathcal{P}_v \subset \mathcal{C}$ such that*

$$V \setminus \{v\} = \bigcup_{C \in \mathcal{P}_v} C \quad \textit{and} \quad C \cap C' = \emptyset \quad \textit{for} \quad C, C' \in \mathcal{P}_v, \quad \textit{whenever} \quad C \neq C'. \tag{A}$$

Let $\mathbb{B} = \{0,1\}$, and let us consider the characteristic vectors $\mathbf{x}^C \in \mathbb{B}^V$ of the sets $C \in \mathcal{C}$, the vector of all ones $\mathbf{e} \in \mathbb{B}^V$, and the unit vectors $\mathbf{e}_v \in \mathbb{B}^V$ for $v \in V$. With this notation we can rewrite condition (A) as

$$\forall v \in V \ \exists \mathcal{P}_v \subset \mathcal{C} \quad \text{such that} \quad \mathbf{x}^{V \setminus \{v\}} = \mathbf{e} - \mathbf{e}_v = \sum_{C \in \mathcal{P}_v} \mathbf{x}^C. \tag{A*}$$

It is clear that the family of ω-cliques in a partitionable (α, ω)-graph is a partitionable hypergraph. The following statement is proving that in fact these are the only partitionable families, or equivalently that one can describe, essentially uniquely, a partitionable graph by specifying the hypergraph of its maximum cliques.

Theorem 1 *Let us assume that $\mathcal{C}$ is a partitionable family of subsets of a finite set V of size $|V| = n$. Then there exists a partitionable (α, ω)-graph $G = (V, E)$ in which $\mathcal{C}$ is the family of maximum cliques. Moreover, $n = \omega\alpha + 1$ and the parameters ω, α as well as the family of maximum stable sets of G are all determined uniquely by $\mathcal{C}$.*

Proof. Obviously, the vectors $\{\mathbf{e} - \mathbf{e}_v \mid v \in V\}$ form a basis in $\mathbb{R}^V$. If the family $\mathcal{C}$ is partitionable then by (A*) every such vector is a linear combination (with binary coefficients) of some of the vectors $\mathbf{x}^C$, $C \in \mathcal{C}$, implying that the set of vectors $\{\mathbf{x}^C \mid C \in \mathcal{C}\}$ is a generator of $\mathbb{R}^V$. Since $|\mathcal{C}| \leq |V|$ is also assumed, we immediately can arrive to the following consequences:

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$$|\mathcal{C}| = |V|, \tag{1a}$$
$$\text{the vectors} \quad \mathbf{x}^C \quad \text{for} \quad c \in \mathcal{C} \quad \text{form a basis of} \quad \mathbb{R}^V, \quad \text{and} \tag{1b}$$
$$\text{the partition} \quad \mathcal{P}_v \subset \mathcal{C} \quad \text{is unique for every} \quad v \in V. \tag{1c}$$

Let us now fix a set $C \in \mathcal{C}$ and let us sum up the equations of (A*) for $v \in C$. We obtain

$$\sum_{v \in C} (\mathbf{e} - \mathbf{e}_v) = |C|\mathbf{e} - \mathbf{x}^C = \sum_{v \in C} \sum_{C' \in \mathcal{P}_v} \mathbf{x}^{C'}$$

from which we can express $\mathbf{e}$ as

$$\mathbf{e} = \frac{1}{|C|} \left(\mathbf{x}^C + \sum_{v \in C} \sum_{C' \in \mathcal{P}_v} \mathbf{x}^{C'} \right). \tag{2}$$

According to (1b), the vectors from the right hand side of (2) form a basis of $\mathbb{R}^V$, hence the expression in (2) must be the unique representation of $\mathbf{e}$ in the basis $\{\mathbf{x}^C \mid C \in \mathcal{C}\}$. Since $C \notin \mathcal{P}_v$ for any $v \in C$ by definition, we obtain that the coefficient of $\mathbf{x}^C$ in the unique representation of $\mathbf{e}$ must be equal to $\frac{1}{|C|}$, for all $C \in \mathcal{C}$. On the other hand, looking at (2) for a fixed set $C \in \mathcal{C}$, we can observe that for any other set $C' \in \mathcal{C}$, the coefficient of the vector $\mathbf{x}^{C'}$ on the right hand side is an integer multiple of $\frac{1}{|C|}$. In other words, the coefficient of $\mathbf{x}^{C'}$ can be equal to $\frac{1}{|C'|}$ only if all sets appear exactly once on the right hand side of (2) and all sets $C \in \mathcal{C}$ have the same size. Let us denote this common size of the sets in $\mathcal{C}$ by ω. It follows then that all the partitions $\mathcal{P}_v$ for $v \in V$ are of the same cardinality, which we shall denote by α. Thus, we can draw the following chain of conclusions:

$$|C| = \omega \quad \text{for all} \quad C \in \mathcal{C}, \quad |\mathcal{P}_v| = \alpha \quad \text{for all} \quad v \in V, \quad \text{and} \quad n = \alpha\omega + 1; \tag{3a}$$
$$\text{every point} \quad v \in V \quad \text{belongs to exactly} \quad \omega \quad \text{sets} \quad C \in \mathcal{C}; \tag{3b}$$

and for every $C \in \mathcal{C}$

$$\text{the families} \quad \mathcal{P}_v,\ v \in C, \quad \text{are pairwise disjoint and form a partition of} \quad \mathcal{C} \setminus \{C\}. \tag{3c}$$

We can also rewrite (3c) as

$$\forall C, C' \in \mathcal{C}, C \neq C', \quad \exists^! v \in C \setminus C' \quad \text{such that} \quad C' \in \mathcal{P}_v. \tag{3c'}$$

From this, by a simple counting argument we can conclude that

$$\text{every set } C \in \mathcal{C} \quad \text{belongs to exactly} \quad \alpha \quad \text{partitions} \quad \mathcal{P}_v, v \in V. \tag{3d}$$

To verify (3d), let us introduce the notation

$$S_C = \{v \in V \mid C \in \mathcal{P}_v\} \tag{4}$$

for $C \in \mathcal{C}$. Clearly, $C \cap S_C = \emptyset$, by the definition. On the other hand, the set C must belong to exactly one of the partitions $\mathcal{P}_v$, $v \in C'$ for any other set $C' \in \mathcal{C}$, $C' \neq C$ by (3c), implying thus

$$C \cap S_C = \emptyset \quad \text{and} \quad |C' \cap S_C| = 1 \quad \text{for all} \quad C, C' \in \mathcal{C},\ C \neq C'. \tag{5}$$

Since a partition $\mathcal{P}_v$ for any $v \in C$ contains α pairwise disjoint sets $C' \neq C$, $|S_C| \geq \alpha$ is implied by (5). By counting the pairs $C \in \mathcal{P}_v$ first by $v \in V$, and second by $C \in \mathcal{C}$, we obtain

$$\sum_{v \in V} |\mathcal{P}_v| = \sum_{C \in \mathcal{C}} |S_C|.$$

from this, using (3a) and the lower bound on $|S_C|$, we get

$$n\alpha = \sum_{v \in V} |\mathcal{P}_v| = \sum_{C \in \mathcal{C}} |S_C| \geq n\alpha,$$

which implies the equality

$$|S_C| = \alpha \quad \text{for all} \quad C \in \mathcal{C}, \tag{6}$$

proving hence (3d).

Remark 1 *Formula (4) is especially important for our approach. Given a partitionable family $\mathcal{C}$, we* introduce *a family $\mathcal{S}$ by formula (4), and then prove that this new family is partitionable, too. Bland, Huang and Trotter (1979) introduce $\mathcal{C}$ and $\mathcal{S}$ together and then define partitionability in terms of both families.*

Let us prove now that the family $\mathcal{S} = \{S_C \mid C \in \mathcal{C}\}$ forms a partitionable family of α-sets. For this we claim that subfamily

$$\mathcal{Q}_v = \{S_C \mid C \in \mathcal{C}, C \ni v\}$$

is a partition of $V \setminus \{v\}$, for every $v \in V$.

Let us note first that if $v \in S_C \cap S_{C'}$, then by (4) both sets C and C' belong to the partition $\mathcal{P}_v$, and hence either $C = C'$, or $C \cap C' = \emptyset$. Thus, we get

$$S_C \cap S_{C'} = \emptyset \text{ whenever } C \cap C' \neq \emptyset \text{ and } C \neq C'. \tag{7}$$

This implies that the sets $S_C \in \mathcal{Q}_v$ are pairwise disjoint. Since $v \notin S_C$ for $S_C \in \mathcal{Q}_v$ by definition, and $|\mathcal{Q}_v| = \omega$ by (3b), the subfamily $\mathcal{Q}_v$ forms a partition of a subset of $V \setminus \{v\}$ of size $\alpha\omega = n-1$, i.e. it forms a partition of $V \setminus \{v\}$.

We can now define a partitionable graph $G = G(\mathcal{C}, \mathcal{S})$ on the vertex set $V(G) = V$, in which the sets $C \in \mathcal{C}$ are the ω-cliques, and the sets $S \in \mathcal{S}$ are the α-stable sets. In other words, for any two distinct vertices $u, v \in V$, let us say that $(u, v) \in E(G)$ if $u, v \in C$ for some $C \in \mathcal{C}$ and $(u, v) \notin E(G)$ if $u, v \in S$ for some $S \in \mathcal{S}$. We do not get any contradiction in this way, since $|C \cap S| \leq 1$ for all $C \in \mathcal{C}$ and $S \in \mathcal{S}$, according to (5). Yet, the graph $G(\mathcal{C}, \mathcal{S})$ is not well defined, because there can be pairs of vertices which do not belong neither to ω-cliques nor to α-stable sets. Such pairs of vertices are called *indifferent edges*. An arbitrary subset of indifferent edges can be included in the graph $G(\mathcal{C}, \mathcal{S})$. Thus in fact, $G(\mathcal{C}, \mathcal{S})$ is not one graph but a family of (equivalent partitionable) graphs. It was proved by Bland, Huang and Trotter (1979) that each of these graphs has exactly n cliques $C \in \mathcal{C}$ of cardinality ω and exactly n stable sets $S \in \mathcal{S}$ of cardinality α; moreover, there are no cliques of cardinality $\omega + 1$, unless $\omega = n - 1$, and similarly, there are no stable sets of cardinality $\alpha + 1$, unless $\alpha = n - 1$. □

Remark 2 *In principle, partitionable families could have parameters $(\alpha, \omega) = (1, n-1)$ or $(\alpha, \omega) = (n-1, 1)$. However, when dealing with partitionable graphs the standard assumption is that $\alpha > 1$ and $\omega > 1$.*

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2.2 Geometric axioms

The following nice "projective" approach to partitionability was suggested by Michael Temkin (private communications). Given a set $V = \{v_1, ..., v_n\}$ and two families of its subsets $\mathcal{C} = \{C_1, ..., C_n\}$ and $\mathcal{S} = \{S_1, ..., S_n\}$ such that $C_1 \cap S_1 = \emptyset, ..., C_n \cap S_n = \emptyset$, let us introduce a *projective biplane* whose n points are $v_1, ..., v_n$ and n lines are $L_1 = C_1 \cup S_1, ..., L_n = C_n \cup S_n$. The difference between the standard finite projective plane and biplane is as follows. The incidence function $F(L_i, v_j)$ for a standard plane takes two values: $F(L_i, v_j) = 1$ if $v_j \in L_i$ and $F(L_i, v_j) = 0$ if $v_j \notin L_i$, while for a biplane it takes three values: $F(L_i, v_j) = 1$ if $v_j \in C_i$, $F(L_i, v_j) = -1$ if $v_j \in S_i$, and $F(L_i, v_j) = 0$ if $v_j \notin L_i$.

Also the intersection of lines $L_i = C_i \cup S_i$ and $L_j = C_j \cup S_j$ is understood in a rather unusual way: $L_i \cap L_j \doteq (C_j \cap S_i) \cup (C_i \cap S_j)$, that is only the points which belong to both lines and whose incidence functions with respect to these two lines have opposite signs are included in the intersection, while the points from $(C_i \cap C_j) \cup (S_i \cap S_j)$ do not count. After the above two innovations a finite projective biplane is defined by the following two more or less standard axioms:

$$\begin{gathered}\text{Every two different lines } L_i = C_i \cup S_i \text{ and } L_j = C_j \cup S_j \\ \text{intersect in exactly two different points } v_k \text{ and } v_m \\ \text{such that } v_k \in C_i \cap S_j \text{ and } v_m \in C_j \cap S_i;\end{gathered} \tag{G1}$$

$$\begin{gathered}\text{Every two different points } v_k \text{ and } v_m \\ \text{are connected by exactly two different lines} \\ L_i = C_i \cup S_i \text{ and } L_j = C_j \cup S_j \text{ such that } v_k \in C_i \cap S_j \text{ and } v_m \in C_j \cap S_i.\end{gathered} \tag{G2}$$

Let us prove that axioms ((G1), (G2)) and (A1) are equivalent. First, given a set $V = \{v_1, ..., v_n\}$ and a partitionable (i.e. satisfying (A1)) family $\mathcal{C} = \{C_1, ..., C_n\}$, let us generate the family $\mathcal{S} = \{S_1, ..., S_n\}$, according to (3d), consider the corresponding biplane and prove that ((G1), (G2)) hold. Formula (G1) results directly from (4). To prove (G2) let us fix any two different points $v_k, v_m \in V$ and consider all the ω sets $C_j, j \in J(v_k)$ which contain v_m, see (3c). According to (3d), the corresponding ω sets $S_j, j \in J(v_k)$ are pairwise disjoint and each one contains α points, according to (5). Hence, together they contain $n-1$ points and must form a partition $\mathcal{P}(v_m)$, that is exactly one of these sets, let us say S_{j_0}, contains v_k. Thus, there exists a unique $j_0 \in [n]$ such that $v_m \in C_{j_0}$ and $v_k \in S_{j_0}$. In the same way we prove that there exists a unique i_0 such that $v_m \in S_{i_0}$ and $v_k \in C_{i_0}$. Thus, (G2) holds.

Now let us derive (A1) from ((G1), (G2)). That is given a biplane, let us prove that family $\mathcal{C} = \{C_1, ..., C_n\}$ must be partitionable. For this let us fix an arbitrary point $v \in V$ and consider all the lines $L_j = C_j \cup S_j$ $j \in J(v)$ such that $v \in S_j$. Then (7) implies that $\mathcal{P}(v) = \{C_j,\ j \in J(v)\}$ is a partition of $V \setminus \{v\}$.

2.3 Matrix axioms

The following matrix approach to partitionability was suggested by Chvátal, Graham, Perold and Whitesides (1979). Let us consider the equation

$$XY = J - I \tag{M}$$

in $n \times n$ (0,1)-matrices where I is the identity matrix, J is the matrix whose all n^2 entries are 1's, and X, Y are unknown.

Again, given a set $V = \{v_1, ..., v_n\}$ and two arbitrary families of its subsets $\mathcal{C} = \{C_1, ..., C_n\}$ and $\mathcal{S} = \{S_1, ..., S_n\}$, let us introduce X as the incidence matrix of V (columns) and $\mathcal{C}$ (rows), and Y as the incidence matrix of V (rows) and $\mathcal{S}$ (columns). Vice versa, to any two binary $n \times n$ matrices X and Y we can associate

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a set V and two families $\mathcal{C}$ and $\mathcal{S}$ of its subsets such that the same incidence relations take place. Thus we get two mutually inverse one-to-one mappings. Let us prove that axioms (M) for X, Y and (A) for $V, \mathcal{C}$ are equivalent. First, (M) is an obvious consequence of (5) because for binary vectors the cardinality of their intersection and their scalar product are the same. Second, (M) implies partitionability of the corresponding set-family $\mathcal{C}$. Indeed, on the one hand, the rows of the matrix $J - I$ are the vectors $e - e_i$ for $i = 1, ..., n$. On the other hand, rows of the matrix product XY are linear combinations of the rows of X, in which all the coefficients take only values 0 or 1. Thus, these linear combinations are simply sums over a subset of the indices. But a sum of characteristic vectors is $e - e_i$ if and only if the corresponding sets from $\mathcal{C}$ form a partition $\mathcal{P}(v_i)$ of $V \setminus \{v_i\}$.

Let us recall that the partitionability of $\mathcal{C}$ implies the partitionability of $\mathcal{S}$. Thus $XY = J - I$ iff $YX = J - I$. Then, let us note next that the matrix $J - I$ is symmetric. This implies $XY = J - I$ iff $Y^t X^t = J - I$, where t indicates matrix transposition. Thus, the following four matrix products XY, YX, $Y^t X^t$, and $X^t Y^t$ can be equal to $J - I$ only simultaneously. If a pair of matrices (X, Y) generates a partitionable graph G then the pair (Y, X) generates the complementary graph G^c, while the pair of the transposed matrices (X^T, Y^T) generates the *dual* partitionable graph G^d (introduced in [12]). Obviously, $G^{dc} = G^{cd}$.

3 Tree-covering families

In this section we study hypergraphs arising by the intersection of a maximum clique in a partitionable graph with all other maximum cliques.

Definition 2 *Let C be a finite set of size ω, and let $\mathcal{A}$ be a family of subsets of C (more precisely, a* multi-family, *i.e. sets in $\mathcal{A}$ may have a multiplicity > 1.) Let us call $\mathcal{A}$ a* tree-covering family, *if*

$$A \in \mathcal{A} \Longrightarrow \overline{A} := C \setminus A \in \mathcal{A}, \tag{C1}$$

and if for every point $v \in C$ there is a subfamily $\mathcal{R}_v \subset \mathcal{A}$ which forms a partition of $C \setminus \{v\}$, i.e. if

$$\forall \quad v \in C \qquad \exists \quad \mathcal{R}_v \subset \mathcal{A} \text{ such that } C \setminus \{v\} = \biguplus_{A \in \mathcal{R}_v} A, \tag{C2}$$

where $\biguplus$ denotes "disjoint union".

Using the characteristic vectors $\mathbf{x}^A \in \mathbb{B}^C$, $A \in \mathcal{A}$, the vector of all ones $\mathbf{e} \in \mathbb{B}^C$, and the unit vectors $\mathbf{e}_v \in \mathbb{B}^C$ for $v \in C$, conditions (C1) and (C2) can be equivalently restated as

$$\forall A \in \mathcal{A} \quad \exists \overline{A} \in \mathcal{A} \quad \text{ such that } \quad \mathbf{x}^A + \mathbf{x}^{\overline{A}} = \mathbf{e} \tag{C1*}$$

$$\forall v \in C \quad \exists \mathcal{R}_v \subset \mathcal{A} \quad \text{ such that } \quad \sum_{A \in \mathcal{R}_v} \mathbf{x}^A = \mathbf{e} - \mathbf{e}_v \tag{C2*}$$

Let us show first that a tree-covering family must have at least $2\omega - 2$ elements.

Lemma 1 *Let $\mathcal{A}$ be a tree-covering family on a finite set C of size ω, and let k denote the number of different sets in $\mathcal{A}$. Then $k \geq 2\omega - 2$.*

Proof. Let us observe first that k is even, since the different sets of $\mathcal{A}$ can be divided into complementary pairs by (C1). Let us denote these complementary pairs by A_i, $\overline{A}_i$, $i = 1, ..., \frac{k}{2}$.

Let us next observe that by (C2*) all vectors of the form $\mathbf{e} - \mathbf{e}_v$ for $v \in C$ can be expressed as linear combinations of the vectors $\mathbf{x}^A$, $A \in \mathcal{A}$. Since the set of vectors $\{\mathbf{e} - \mathbf{e}_v \mid v \in C\}$ forms a basis of $\mathbb{R}^C$, the set $\{\mathbf{x}^{A_i}, \mathbf{x}^{\overline{A}_i} \mid i = 1, ..., \frac{k}{2}\}$ must span $\mathbb{R}^C$. Let us now consider a subfamily, $B = \{\mathbf{x}^{A_i} \mid i = 1, ..., \frac{k}{2}\} \cup \{\mathbf{x}^{\overline{A}_1}\}$ consisting of the first complementary pair, and one of the characteristic vectors for all other complementary pairs. According to (C1*), we can obtain all other characteristic vectors by $\mathbf{x}^{\overline{A_i}} = (\mathbf{x}^{A_1} + \mathbf{x}^{\overline{A}_1}) - \mathbf{x}^{A_i}$ for $i > 1$, and hence B spans $\mathbb{R}^C$, too, implying $|B| \geq \omega$. Since $|B| = 1 + \frac{k}{2}$, the statement of the lemma follows immediately. □

Definition 3 *Let us call a tree-covering family $\mathcal{A}$ on a finite set C of size ω* critical*, if it has the smallest possible size, i.e. if*

$$|\mathcal{A}| = 2\omega - 2. \tag{C3}$$

An immediate corollary from the proof of Lemma 1 is that all sets of a critical tree-covering family must have a multiplicity of 1. Thus, since in the sequel we shall talk about critical tree-covering families, we do not have to pay special attention to distinguishing families from multi-families.

Let us see first some examples for critical tree-covering families: Let us consider an arbitrary spanning tree T on the vertex set $V(T) = C$ of cardinality ω. The removal of an edge $(u, v) \in E(T)$ divides the set of vertices into two connected components. Let us denote the component containing v but not u by A_{uv} and let A_{vu} be the other component. Finally, let us define a family $\mathcal{A}_T = \{A_{uv}, A_{vu} \mid (u, v) \in E(T)\}$. Clearly, $\mathcal{A}_T$ has $2\omega - 2$ elements, and $\overline{A}_{uv} = A_{vu}$, i.e. both conditions (C1) and (C3) hold. Furthermore, one can see that for every vertex $u \in C$ the subfamily $\mathcal{R}_u = \{A_{uv} \mid (u, v) \in E(T)\}$ forms a partition of the vertex set $C \setminus \{u\}$, since T is a spanning tree on C. Thus $\mathcal{A}_T$ is a critical tree-covering family for every spanning tree T.

We shall show next that in fact all critical tree-covering families arise in this way.

Theorem 2 *If $\mathcal{A}$ is a critical tree-covering family on a finite set C, then there exists a spanning tree T on C such that $\mathcal{A} = \mathcal{A}_T$.*

To prove this theorem, we shall need a series of simple lemmas first. Let us assume that $\mathcal{A}$ is a critical tree-covering family on a set C of cardinality ω.

Lemma 2 *If*

$$\mathbf{e} = \sum_{A \in \mathcal{A}} \alpha_A \mathbf{x}^A \tag{9}$$

for some nonnegative real coefficients $\alpha_A \geq 0$, $A \in \mathcal{A}$, then there exists a complementary pair of sets, $A \in \mathcal{A}$ and $\overline{A} \in \mathcal{A}$, for which both coefficients α_A and $\alpha_{\overline{A}}$ are positive.

Proof. Let us assume indirectly that $\min(\alpha_A, \alpha_{\overline{A}}) = 0$ for all $A \in \mathcal{A}$, and let us choose a subfamily $\mathcal{B} \subset \mathcal{A}$ by defining

$$\mathcal{B} = \{A \mid \alpha_A > 0\} \cup \{A \mid \alpha_A = \alpha_{\overline{A}} = 0 \text{ and } v \in A\}$$

where $v \in C$ is a fixed element. Clearly, in this way we choose into $\mathcal{B}$ exactly one set from each complementary pair in $\mathcal{A}$. The subfamily $\mathcal{B}$ also contains all sets to which the corresponding vector on the right hand side of (9) has a positive coefficient. Using then (C1*) and (9), we can conclude that the vectors $\mathbf{x}^A$, $A \in \mathcal{B}$ must form a generating set, just like in the proof of Lemma 1. This is a contradiction with the fact that $|\mathcal{B}| = \omega - 1$ for a critical tree-covering family, and hence the lemma follows. □

For a critical tree-covering family $\mathcal{A}$ on the set C, let us choose a subfamily $\mathcal{R}_v$ for every $v \in C$ for which condition (C2) holds (in principle, such subfamilies may not be unique).

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Lemma 3 *For every set $A \in \mathcal{A}$ there exists a unique vertex $v \in C$ such that $A \in \mathcal{R}_v$.*

Proof. By summing up the equations (C2*), we get

$$\sum_{v \in C} \sum_{A \in \mathcal{R}_v} \mathbf{x}^A = (\omega - 1)\mathbf{e}. \tag{10}$$

Let us denote by m^A the number of points $v \in C$ for which $A \in \mathcal{R}_v$, and let $v \in C$ be a fixed vertex. With this notation (10) can be rewritten as

$$\begin{aligned}(\omega - 1)\mathbf{e} &= \sum_{A \in \mathcal{A}} m^A \mathbf{x}^A \\ &= \sum_{A \in \mathcal{A}, v \in A} \min(m^A, m^{\overline{A}}) \left(\mathbf{x}^A + \mathbf{x}^{\overline{A}}\right) + \sum_{A \in \mathcal{A}} \left(m^A - m^{\overline{A}}\right)_+ \mathbf{x}^A.\end{aligned}$$

where $(a-b)_+ = a - b$ if $a > b$, and $(a-b)_+ = 0$ otherwise. Using (C1*), we obtain finally

$$\left[(\omega - 1) - \sum_{A \in \mathcal{A}, v \in A} \min(m^A, m^{\overline{A}})\right] \mathbf{e} = \sum_{A \in \mathcal{A}} \left(m^A - m^{\overline{A}}\right)_+ \mathbf{x}^A. \tag{11}$$

The right hand side above is componentwise nonnegative, hence $\sum_{A \in \mathcal{A}, v \in A} \min(m^A, m^{\overline{A}}) \leq \omega - 1$ follows. If the left hand side of (11) were in fact non zero, we could obtain from (11) the vector $\mathbf{e}$ as a nonnegative combination of the vectors $\mathbf{x}^A$, $A \in \mathcal{A}$. According to Lemma 2 this would imply that for at least one set $S \in \mathcal{S}$ both $(m^A - m^{\overline{A}})_+$ and $(m^{\overline{A}} - m^A)_+$ are positive, which is impossible, since for any two reals a and b, either $(a-b)_+ = 0$ or $(b-a)_+ = 0$ (or both). This contradiction shows that

$$\omega - 1 = \sum_{A \in \mathcal{A}, v \in A} \min(m^A, m^{\overline{A}}). \tag{12}$$

Thus all the nonnegative coefficients on the right hand side of (11) must also be equal to zero, that is $(m^A - m^{\overline{A}})_+ = 0 = (m^{\overline{A}} - m^A)_+$ implying thus $m^A = m^{\overline{A}}$ for all $A \in \mathcal{A}$.

Let us observe next that $m^A > 0$ for all $A \in \mathcal{A}$, since otherwise we have $m^A = m^{\overline{A}} = 0$ for some sets $A \in \mathcal{A}$, implying that the family $\mathcal{A}' = \mathcal{A} \setminus \{A, \overline{A}\}$ is again a tree-covering family of size $|\mathcal{A}| - 2 < 2\omega - 2$, a contradiction to Lemma 1.

Since in the summation of the right hand side of (12) we have $\omega - 1$ terms, and since each of those is a nonnegative integer according to the above, we can conclude from (12) that $m^A = 1$ for all $A \in \mathcal{A}$, hence proving the lemma. □

The above lemma shows also that in a critical tree-covering family $\mathcal{A}$ for every vertex $v \in C$ there is a unique subfamily $\mathcal{R}_v \subset \mathcal{A}$ which forms a partition of the vertex set $C \setminus \{v\}$.

Let us now consider a graph T on the vertex set $V(T) = C$ with an edge set defined by

$$E(T) = \{(u,v) \mid u, v \in C, \quad \text{and} \quad \exists A \in \mathcal{A} \quad \text{such that} \quad A \in \mathcal{R}_v \text{ and } \overline{A} \in \mathcal{R}_u\}.$$

Since a critical tree-covering family $\mathcal{A}$ consists of $\omega - 1$ complementary pairs, it follows by Lemma 3 that the graph T has exactly $\omega - 1$ edges, one corresponding to each complementary pair of sets of $\mathcal{A}$. For an edge $(u,v) \in E(T)$ let us denote the corresponding complementary sets of $\mathcal{A}$ by A_{uv} and $A_{vu} = \overline{A}_{uv}$ such that $v \in A_{uv}$ and $u \in A_{vu}$. It is easy to see that Lemma 3 and the above definitions readily imply

Corollary 1 *There are no loops in T, and we have $\mathcal{A} = \{A_{uv}, A_{vu} \mid (u,v) \in E(T)\}$.* □

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Lemma 4 *For every $v \in C$ we have $\mathcal{R}_u = \{A_{uv} \,|\, (u,v) \in E(T)\}$.*

Proof. The relation $\mathcal{R}_u \supseteq \{A_{uv} \,|\, (u,v) \in E(T)\}$ follows directly from the definition of the edges of T.

For the converse relation, let $A \in \mathcal{R}_u$ be arbitrary. Then $\overline{A} \in \mathcal{A}$ by (C1), and thus by Lemma 3 there exists a unique vertex $v \in C$ for which $\overline{A} \in \mathcal{R}_v$. Clearly $u \neq v$, since $u \in \overline{A}$ and $\overline{A} \subseteq C \setminus \{v\}$. Therefore, $(u,v) \in E(T)$ and $A = A_{uv}$ follows by the definition of T. □

Lemma 5 *If $(u,v) \in E(T)$ and $(v,w) \in E(T)$, then $A_{uv} \subset A_{vw}$.*

Proof. According to Lemma 4 we have $A_{vw} \in \mathcal{R}_v$ and $A_{vu} \in \mathcal{R}_v$, thus $A_{vw} \cap A_{vu} = \emptyset$. Since $\overline{A}_{uv} = A_{vu}$, we get $A_{uv} \supseteq A_{vw}$, where all relations are strict containments, because $v \in A_{uv}$, while $v \notin A_{vw}$. □

Lemma 6 *There are no circuits in T.*

Proof. Let us assume indirectly that u_1, ..., u_k are vertices from C forming a cycle, i.e. $(u_i, u_{i+1}) \in E(T)$ for $i = 1, ..., k-1$, and $(u_k, u_1) \in E(T)$. Then, by Lemma 5 we would have $A_{u_1u_2} \supset A_{u_2u_3} \supset \cdots \supset A_{u_ku_1} \supset A_{u_1u_2}$, all relations as strict containment, a clear contradiction, proving the lemma. □

Proof of Theorem 2. The graph T constructed above is a spanning tree on C by Lemma 6, and the equality $\mathcal{A} = \mathcal{A}_T$ follows by Corollary 1 and Lemma 4. □

4 Critical edges and cliques in partitionable graphs

Given a partitionable (α,ω)-graph G on the vertex set V of cardinality n, let $\mathcal{C}$ be the partitionable family of its ω-cliques, and let $\mathcal{P}_v \subset \mathcal{C}$ denote the partition of $V \setminus \{v\}$ for $v \in V$. By (1c) the partition $\mathcal{P}_v$ is unique for every $v \in V$. For every $C \in \mathcal{C}$ we define $S_C = \{v \in V \mid C \in \mathcal{P}_v\}$, as in (4), and $\mathcal{S} = \{S_C \mid C \in \mathcal{C}\}$. It was shown in the proof of Theorem 1 that $\mathcal{S}$ is the family of all maximum stable sets of the graph G, and the set S_C is the unique vis-a-vis stable set of the clique C, for all $C \in \mathcal{C}$.

Given a critical clique C of G, denote by $\mathcal{M}^C = \{\tilde{C} \in \mathcal{C} \mid \tilde{C} \neq C \text{ and } C \cap \tilde{C} \neq \emptyset\}$ the family of maximum cliques of G intersecting C. Further let $\mathcal{A}^C = \{\tilde{C} \cap C \mid \tilde{C} \in \mathcal{M}^C\}$.

Clearly, the partitionability of $\mathcal{C}$ and the the fact that C is a critical clique, that is that $|\mathcal{M}^C| = 2\omega - 2$, implies that $\mathcal{A}^C$ is a critical tree-covering family. Hence, by Theorem 2, we can conclude that $\mathcal{A}^C = \mathcal{A}_T$ for some spanning tree $T = T_C$ of the vertex set C, that is $\mathcal{A}^C = \{A_{uv}, A_{vu} \mid (u,v) \in E(T)\}$, where $E(T)$ denotes the edge set of T, and $A_{uv} \cup A_{vu} = C$, $u \in A_{vu}$, $v \in A_{uv}$, $A_{uv} \cap A_{vu} = \emptyset$ for all edges $(u,v) \in E(T)$, exactly as in the previous section. Let us then denote by $C_{uv} \in \mathcal{M}^C$ the maximum clique for which $A_{uv} = C \cap C_{uv}$, for $(u,v) \in E(T)$. Then, by (4), it follows for every edge $(u,v) \in E(T)$ that $u \in S_{C_{uv}}$, $v \in S_{C_{vu}}$ and $|S_{C_{uv}} \cap S_{C_{vu}}| = \alpha - 1$. This means that the edges $(u,v) \in E(T)$ are critical. Furthermore we have

$$\forall v \in V \setminus (C \cup S_C) \;\; \exists^! (u,v) \in E(T) \quad \text{such that} \quad \{C_{uv}, C_{vu}\} \subseteq \mathcal{P}_v. \tag{13}$$

Thus we reproduced partially the results of Sebő (1996b) cited earlier. Let us point out however that our proof was based only on the combinatorial properties of critical tree-covering hypergraphs and partitionable families. This might suggest that perhaps all critical tree-covering families can arise from partitionable graphs in this way, a possibility not yet confirmed. This question is one of the motivating factors behind this paper. More precisely, can an arbitrary spanning tree T on ω vertices appear as the tree formed by the critical edges in some partitionable (α,ω)-graph? Though this question is still open, the next claim shows that the answer is negative if we fix the value of α as well.

For a tree T let us denote by $L(T)$ the set of its *leaves*, that is vertices of degree 1.

Lemma 7 *Let $T = T_C$ be the spanning tree of critical edges in a critical clique C of a partitionable (α, ω)-graph G. Then, we have*

$$|L(T)| \leq \alpha.$$

Proof. Since every leaf node $v \in C$ of T is incident with exactly one edge of T, the set $C \setminus \{v\}$ arises as the intersection of C with another maximum clique $\tilde{C}_v \in \mathcal{M}^C$. Let u_v denote the unique vertex in $\tilde{C}_v \setminus C$. Since $\tilde{C}_v$ has only one point, namely u_v, outside of C, this vertex must belong to the vis-a-vis stable set S_C, because all cliques (other than C) must intersect S_C. Let us also note that such a vertex u_v is adjacent to all vertices of C other than v. This implies that the vertices u_v and u_w corresponding to two different leaves v and w of T must be different, since otherwise $\{v, u_v\} \subset \tilde{C}_{v_w}$ would imply that $(v, u_v) \in E(G)$, that is the set $C \cup \{u_v\}$ would be an $(\omega + 1)$-clique of G. Thus, $|\{u_v \mid v \in L(T)\}| = |L(T)|$ and $\{u_v \mid v \in L(T)\} \subseteq S_C$ both hold, implying hence the claim. □

In particular, the above lemma implies that every vertex of the tree T is incident to at most α critical edges, that is $deg_T(v) \leq \alpha$.

Let us also note that the properties shown in the previous sections and the above analysis also implies the following decomposition of the partitionable graphs which have a critical clique.

Corollary 2 *Let G be a partitionable (α, ω)-graph on the vertex set V, and let $\mathcal{C}$ denote the family of its ω-cliques. Let us assume further that $C \in \mathcal{C}$ is a critical clique of G, and let T be the corresponding tree of critical edges in C, as above. Then the following claims hold:*

- *There are pairwise disjoint subsets $U_e \subseteq V \setminus (C \cup S_C)$, $e \in E(T)$ of size $\alpha - 1$ each (i.e. these subsets form a partition of $V \setminus (C \cup S_C)$) such that for every critical edge $e = (u, v) \in E(T)$ both sets $U_e \cup \{u\}$ and $U_e \cup \{v\}$ are maximum stable sets of G. In fact, these stable sets are the vis-a-vis stable sets of the cliques C_{uv} and C_{vu}, introduced earlier in this section.*
- *There are no other critical edges (other than those in $E(T)$) in C.*
- *Each of the $(\alpha - 2)\omega + 2$ maximum cliques of G not intersecting C contains exactly one-one points of the sets U_e, $e \in E(T)$ and S_C.*
- *The $2\omega - 1$ stable sets $\{U_e \cup \{u\},\ U_e \cup \{v\} \mid u = (u, v) \in E(T)\} \cup \{S_C\}$ include all stable sets needed to form the stable set partitions of $V \setminus \{v\}$ for every $v \in C$. Namely, let us consider a vertex $v \in C$ as the root of T, and let us orient the edges of T away from v. Let us denote by u_e the end nodes of edges $e \in E(T)$. Then*

  $$\mathcal{Q}_v = \{S_C\} \cup \{U_e \cup \{u_e\} \mid e \in E(T)\}$$

  is the unique partition of $V \setminus \{v\}$ by maximum stable sets.

□

Let us finally remark that the sets U_e, $e \in E(T)$, and S_C form that unique coloration of $V \setminus C$ mentioned in the result which we cited earlier from Sebő (1996b).

5 Reduction

Given a partitionable family $\mathcal{C}$ of the ω-cliques of a partitionable (α, ω)-graph G on vertex set V and a critical clique $C \in \mathcal{C}$ of this graph, we shall construct another family $\mathcal{C}'$ on the set $V' = V \setminus C$ and show that $\mathcal{C}'$ is partitionable, too, or in other words that $\mathcal{C}'$ is the family of ω-cliques of a partitionable $(\alpha - 1, \omega)$-graph G' on the vertex set V'.

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Let T be the spanning tree formed by the critical edges in C, and let $E = E(T)$. Let us consider the family

$$\mathcal{M}^C = \{C_e^1, C_e^2 \mid e \in E\}$$

as above. For every $e \in E$ let us define a set

$$C'_e = (C_e^1 \cup C_e^2) \setminus C, \tag{14}$$

and finally, let us define the new family by

$$\mathcal{C}' = \big(\mathcal{C} \setminus (\mathcal{M}^C \cup \{C\})\big) \cup \{C'_e \mid e \in E\}. \tag{15}$$

Theorem 3 *The reduced family $\mathcal{C}'$ is a partitionable family on the set $V' = V \setminus C$.*

Proof. All sets in $\mathcal{C}'$ are subsets of V' by the definition, and we have

$$|\mathcal{C}'| = |\mathcal{C}| - (|\mathcal{M}^C| + 1) + |E| = n - (2\omega - 1) + (\omega - 1) = n - \omega = |V \setminus C| = |V'|.$$

Thus, to prove the theorem it is enough to show by Theorem 1 that for every $v \in V'$ there exists a subset $\mathcal{P}'_v \subset \mathcal{C}'$ partitioning the set $V' \setminus \{v\}$.

Let us consider first the family $\mathcal{P}_v \subset \mathcal{C}$. If $C \in \mathcal{P}_v$, then $\mathcal{P}_v \cap \mathcal{M}^C = \emptyset$, and thus

$$\mathcal{P}'_v = \mathcal{P}_v \setminus \{C\}$$

is a desired partition within $\mathcal{C}'$. On the other hand, if $C \notin \mathcal{P}_v$, then $v \in V \setminus (C \cup \mathcal{S}_C)$, and thus by (13) there exists a unique $e \in E$ such that

$$\mathcal{P}_v \cap \mathcal{M}^C = \{C_e^1, C_e^2\}.$$

In this case the family

$$\mathcal{P}'_v = \big(\mathcal{P}_v \setminus \{C_e^1, C_e^2\}\big) \cup \{C'_e\}$$

will be a subfamily of $\mathcal{C}'$ partitioning the set $V' \setminus \{v\}$. $\square$

6 Recursion

To be able to find a constructive inverse to the above reduction operation, let us first analyze the structure of the restrictions of the hypergraph $\mathcal{C}$ to the sets C and $V \setminus C$, separately. Let us observe first that

$$\text{The family } \mathcal{A}^C = \{C \cap \tilde{C} | \tilde{C} \in \mathcal{M}^C\} \text{ is a critical tree-covering family.} \tag{R1}$$

Clearly, conditions (C1) and (C3) hold, because C is a critical clique. To see (C2), let us define

$$\mathcal{R}_v = \{C \cap \tilde{C} \mid \tilde{C} \in \mathcal{M}^C \cap \mathcal{P}_v\}$$

for every $v \in C$. Then, $\mathcal{R}_v \subset \mathcal{A}$, and its members form a partition of the set $C \setminus \{v\}$ by the definition and by the fact that $\mathcal{C}$ is a partitionable family.

Let $T = T_C$ denote again the spanning tree of critical edges in C and let $E = E(T)$ be the edge set of T.

Let us note next that the family $\mathcal{B} = \{C'_e | e \in E\}$ is a subfamily of $\mathcal{C}'$ of cardinality $\omega - 1$ such that

$$|\mathcal{B} \cap \mathcal{P}'_v| \leq 1 \text{ for all } v \in V', \tag{R2}$$

following immediately by the proof of Theorem 3.

Let us note also that sets in $\mathcal{B}$ are split into two by the sets $\tilde{C} \setminus C$ for $\tilde{C} \in \mathcal{M}^C$ such that

$$\forall v \in C \text{ the set } \left(V' \setminus \bigcup_{\tilde{C} \in \mathcal{P}_v \cap \mathcal{M}^C} (\tilde{C} \setminus C)\right) \text{ is partitioned by } \mathcal{C}'. \tag{R3}$$

Indeed, the sets in $\mathcal{P}_v \cap \mathcal{C}'$ for $v \in C$ provide such a partition.

Remark 3 *Condition (R1) can be restated, due to the results in Section 3, as* $\mathcal{A}^C = \mathcal{A}_T$ *for some spanning tree* T *on the vertex set* C.

Remark 4 *Condition (R2) can also be stated in a different way, by (4), saying that the vis-a-vis stable sets* $S_{C'_e}$ *for* $e \in E$ *are pairwise disjoint. In particular, (R2) holds according to (7), if all* $\omega-1$ *sets* $\{C'_e \mid e \in E(T)\}$ *have a vertex in common, in which case the resulting partitionable graph can be shown, using the properties in Corollary 2 and Corollary 1.2 of Sebő (1996b), to be either an odd hole, or an odd anti-hole.*

Remark 5 *Condition (R3) holds automatically if vertex* $v \in C$ *is a leaf of* T. *This condition could also be translated in terms of the vis-a-vis stable sets* S_C, *as well as in terms of the dual partitionable graph* G^d.

We are now ready to show that the above conditions (R1), (R2) and (R3) are essentially the necessary and sufficient conditions one needs to inverse the reduction.

Yet, we should strengthen (R3) slightly. Let us assume that we are given a partitionable family $\mathcal{C}'$ of ω-sets on the vertex set V', corresponding to a partitionable (α,ω)-graph G'. Let C be a set of size ω, disjoint from V', and let T be a spanning tree on C with edge set $E = E(T)$. Let us denote by T_{uv} and T_{vu} the vertex sets of the connected components obtained by removing the edge $(u,v) \in E(T)$ from the tree T, such that $v \in T_{uv}$ and $u \in T_{vu}$. Let finally Γ_v denote the set of neighbors of v in T, i.e. $\Gamma_v = \{u \mid (u,v) \in E(T)\}$. Let us further assume that there is a subfamily $\mathcal{B} = \{C'_{uv} | (u,v) \in E(T)\} \subset \mathcal{C}'$ satisfying condition (R2) the cliques of which can be split into two parts $C'_{uv} = B_{uv} \cup B_{vu}$ for $(u,v) \in E(T)$ in such a way that $B_{uv} \cap B_{vu} = \emptyset$, $|B_{uv}| = |T_{uv}|$ (and hence $|B_{vu}| = |T_{vu}|$), and such that

$$\forall v \in C \quad \begin{array}{c} \text{the sets } B_{uv} \text{ for } u \in \Gamma_v \text{ are pairwise disjoint, and} \\ \exists \mathcal{H}_v \subset \mathcal{C}' \setminus \mathcal{B} \text{ partitioning } V' \setminus \bigcup_{u \in \Gamma_v} B_{uv}. \end{array} \tag{R3*}$$

Let us then define

$$\mathcal{C} = (\mathcal{C}' \setminus \mathcal{B}) \cup \{T_{uv} \cup B_{vu}, T_{vu} \cup B_{uv} | (u,v) \in E(T)\} \cup \{C\}. \tag{16}$$

Theorem 4 *The family* $\mathcal{C}$ *is a partitionable family of* ω*-cliques of a partitionable* $(\alpha+1,\omega)$*-graph* G *on the vertex set* $V = V' \cup C$. *Furthermore,* $C \in \mathcal{C}$ *is a critical clique, for which if we apply the reduction, we obtain* $\mathcal{C}'$ *back.*

Proof. Clearly, $\mathcal{C}$ is a family of size

$$|\mathcal{C}| = |\mathcal{C}'| - |\mathcal{B}| + 2|E(T)| + 1 = |\mathcal{C}'| + \omega = |V'| + |C| = |V|.$$

Thus, to prove the first half of the theorem, we need to show that for every $v \in V$ there exists a subfamily of $\mathcal{C}$ partitioning the set $V \setminus \{v\}$.

Let us consider first points $v \in V'$. If $\mathcal{P}'_v \cap \mathcal{B} = \emptyset$, then

$$\mathcal{P}_v = \mathcal{P}'_v \cup \{C\}$$

is an appropriate partitioning subfamily of $\mathcal{C}$. If $\mathcal{P}'_v \cap \mathcal{B} \neq \emptyset$ then, by our assumptions, there is a unique set C'_{uv} of $\mathcal{B}$ which belongs to $\mathcal{P}'_v$. In this case the family

$$\mathcal{P}_v = (\mathcal{P}'_v \setminus \{C'_{uv}\}) \cup \{T_{uv} \cup B_{vu}, T_{vu} \cup B_{uv}\}$$

is a subfamily of $\mathcal{C}$ partitioning the set $V \setminus \{v\}$.

Let us now define for every point $v \in C$

$$\mathcal{P}_v = \mathcal{H}_v \cup \{B_{uv} \cup T_{vu} | u \in \Gamma_v\}.$$

Clearly $\mathcal{P}_v \subset \mathcal{C}$ by this definition, and the sets in $\mathcal{H}_v$ cover with no overlap the points $V' \setminus \bigcup_{u \in \Gamma_v} B_{uv}$ by (R3*), while the sets $B_{uv} \cup T_{vu}$ for $u \in \Gamma_v$ cover, without any overlap by (R3*), the rest of V' and $C \setminus \{v\}$. Thus, $\mathcal{P}_v$ is a partition of $V \setminus \{v\}$ for every $v \in C$.

Since the only sets of $\mathcal{C}$ intersecting C in a nontrivial way, are those of the form $B_{uv} \cup T_{vu}$ and $B_{vu} \cup T_{uv}$ for $(u, v) \in E(T)$, there are exactly $2\omega - 2$ such sets, and hence C is a critical clique of the family $\mathcal{C}$. It is now a straightforward verification that the conditions (R1), (R2) and (R3*) hold, and the reduction starting with $\mathcal{C}$ and $C \in \mathcal{C}$ will yield $\mathcal{C}'$. □

7 Substituting spiders in webs

From a practical point of view, condition (R1) is well characterized in Section 3, hence equivalently we always can start with a spanning tree on the ω-set C. However, finding $\omega - 1$ cliques in $\mathcal{C}'$ satisfying (R2), and finding a split of each of these cliques satisfying (R3*) is far from being trivial.

Given a partitionable (α, ω)-graph $G' = (V', E')$, and a disjoint ω-set C, let us try to construct a partitionable $(\alpha + 1, \omega)$-graph on the vertex set $V' \cup C$, following the recursion described in the previous section. As we have shown, we must choose first a spanning tree T with $V(T) = C$, and use the critical family defined by its edges in our construction.

An immediate question arises: can we pick any spanning tree T on the set C? Applying Lemma 7 we can conclude that the maximum degree of the vertices in T and the number of leaves certainly cannot exceed $\alpha + 1$. We also know that a simple path can surely arise in this way, since this is the case with a web, in which all cliques are critical.

In this section we show that in fact there is an infinite family of trees (much larger than the family of paths but still very restricted) which can arise as spanning trees in critical cliques, by applying the recursive construction described in the previous section. For this we shall consider (α, ω)-webs and apply the recursion to them starting with a special family of spanning trees.

The (α, ω)-*web*, is the graph $G' = (V', E')$, in which the vertices can be identified with the integers modulo $n = \alpha\omega + 1$, i.e. $V' = \mathbb{Z}_n$, and in which the ω-cliques correspond to consecutive (modulo n) sequences of integers in $\mathbb{Z}_n$.

Let us introduce the notations $\Omega = \{0, 1, ..., \omega - 1\} = \mathbb{Z}_\omega$, $\Lambda = \{1, ..., \alpha\} = \mathbb{Z}_\alpha$, and let us have the convention that arithmetical operations with elements of $\mathbb{Z}_n$ will always be meant modulo n. Furthermore, for a subset $S \subseteq \mathbb{Z}_n$ and an integer $a \in \mathbb{Z}_n$ let us define $a + S = \{a + i | i \in S\}$. The family of ω-cliques of the (α, ω)-web G' then can, more precisely, be described as

$$\mathcal{C}' = \{C'_i = i + \Omega | i \in \mathbb{Z}_n\} \tag{17}$$

while its α-stable sets are

$$\mathcal{S}' = \{S'_i = i + \omega * \Lambda | i \in \mathbb{Z}_n\}. \tag{18}$$

With these definitions, C'_i and S'_i are vis-a-vis for all $i \in \mathbb{Z}_n$.

Now let us define a *spider* as a rooted tree in which only the root vertex can have degree higher than 2. For example, a path is a spider, whichever vertex of it is chosen as the root. Given a spider T, the root r of T is called the spider's *head*, and the vertices of T of degree 1 (leaves) are called the spider's *feet*. (The head may coincide with a foot.) Furthermore, the simple paths connecting the feet to the head are called the spider's *legs*. By definition, the number of legs is equal to the degree d_r of the head r of the spider T.

Theorem 5 *Let us consider an (α, ω)-web $G' = (V', E')$ on $n = \alpha\omega + 1$ vertices, and a spanning spider $T = (C, E)$ with its head at $r \in C$, where C is an ω-set, disjoint from V', and let us assume that $d_r \leq \alpha + 1$. Then the recursion of the previous section can be applied and it results in an $(\alpha + 1, \omega)$-partitionable graph G in which $V' \cup C$ is the vertex set, C is a critical clique, and all edges of the spider T are critical.*

Proof. Let us first identify the vertices of G' with $\mathbb{Z}_n$, as above, and let us introduce coordinates for the vertices of T. Let us number the legs first from 1 to d_r, and then let us associate the pair (k, i) to the vertex $v \in C$, if v belongs to the k-th leg, and v is the i-th vertex counted from the foot of that leg, that is $(k, 1)$

for $k = 1, ..., d_r$ are the spider's feet. Let us note that formally all the pairs $(k, n_k + 1)$ for $k = 1, 2, ..., d_r$ are corresponding to the head of the spider, where n_k denotes the number of vertices on the k-th leg (not counting the head). With these notations, we have

$$\sum_{k=1}^{d_r} n_k = \omega - 1 \tag{19}$$

and that

$$C = \{r\} \cup \{(k,i) | 1 \leq i \leq n_k,\ 1 \leq k \leq d_r\}. \tag{20}$$

To simplify notations, let us also introduce subintervals of $\mathbb{Z}_n$ by defining

$$[a, b) = \{a + j | j = 0, 1, ..., (b - a - 1) \mod n\}.$$

For instance for $n = 11$ we have $[4, 8) = \{4, 5, 6, 7\}$ and $[10, 2) = \{10, 0, 1\}$.

To describe our construction, we need to specify $\omega - 1$ cliques of G' corresponding to the edges of T, and an appropriate split of each of them into two subsets.

With our notation, all the edges of T are of the form $[(k,i),(k,i+1)]$ for some indices $1 \leq k \leq d_r$ and $1 \leq i \leq n_k + 1$. In particular, the edge $[(k, n_k),(k, n_k + 1)]$ is the edge of the k-th leg, incident with the head. Then the sets corresponding to the partitions of C induced by these edges are

$$\begin{aligned} T_{[(k,i),(k,i+1)]} &= \{(l,j) | l \neq k\} \cup \{(k,j) | j \geq i+1\}, \text{ while} \\ T_{[(k,i+1),(k,i)]} &= \{(k,j) | j \leq i\}, \end{aligned} \tag{21}$$

for $i = 1, ..., n_k$, and $k = 1, ..., d_r$. Clearly, $|T_{[(k,i+1),(k,i)]}| = i$ and $|T_{[(k,i),(k,i+1)]}| = \omega - i$ for all $1 \leq i \leq n_k$ and $1 \leq k \leq d_r$.

Let us now define the associated ω-cliques of G' by

$$\begin{aligned} C'_{[(k,i),(k,i+1)]} &= [k\omega - (n_1 + \cdots n_{k-1} + i), (k+1)\omega - (n_1 + \cdots n_{k-1} + i)) \\ &= C'_{k\omega-(n_1+\cdots n_{k-1}+i)} \end{aligned} \tag{22}$$

using our notation of (17), for $i = 1, 2, ..., n_k$ and for $k = 1, ..., d_r$. Let us split each of these cliques into two subintervals given by

$$\begin{aligned} B_{[(k,i),(k,i+1)]} &= [k\omega - (n_1 + \cdots n_{k-1}), (k+1)\omega - (n_1 + \cdots n_{k-1} + i)) \text{ and} \\ B_{[(k,i+1),(k,i)]} &= [k\omega - (n_1 + \cdots n_{k-1} + i), k\omega - (n_1 + \cdots n_{k-1})), \end{aligned} \tag{23}$$

We claim that with these definitions, the clique family $\mathcal{C}$, given as in (16), will indeed define an $(\alpha+1, \omega)$-partitionable graph on the vertex set $V' \cup C$. In order to see this, according to Theorem 4, we have to verify that conditions (R1), (R2) and (R3*) are all satisfied by our construction.

The first condition (R1), as we noted earlier, follows directly from the fact that T is a spanning tree, and the splits $T_{[(k,i),(k,i+1)]}$ and $T_{[(k,i+1),(k,i)]}$ are defined by the edges of this tree. Hence, by Theorem 2, they form indeed a critical tree-covering family on C.

To verify condition (R2), we have to show that the cliques $C'_{k\omega-(n_1+\cdots n_{k-1}+i)}$ for $i = 1, 2, ..., n_k$ and for $k = 1, ..., d_r$ all belong to different partitions $\mathcal{P}'_v$ of the (α, ω)-web G'. To this end, let us observe first that, due to the special structure of a web, two cliques C'_i and C'_j $(i < j)$, as defined by (17), belong to the same partition if and only if $j - i \geq \omega$ and $j - i = 0$ or $1 \mod \omega$, i.e. if they do not overlap, and one of the gaps between these two subintervals of the circular $\mathbb{Z}_n$ can be tiled by ω-intervals. Let us now consider two cliques of the form $C'_{k\omega-(n_1+\cdots n_{k-1}+i)}$ and $C'_{k'\omega-(n_1+\cdots n_{k'-1}+i')}$, as in (22). Let us observe that if $k = k'$, then these cliques overlap, and thus cannot belong to the same partition, while for $k > k'$ we have

$$\begin{aligned} (k\omega - (n_1 + \cdots n_{k-1} + i)) &- (k'\omega - (n_1 + \cdots n_{k'-1} + i')) \\ &= (k - k')\omega - (n_{k'} + \cdots n_{k-1} + i - i'). \end{aligned}$$

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Since $n_{k'} - i' \geq 0$, $i \geq 1$ and $k > k'$, the sum $n_{k'} + \cdots n_{k-1} + i - i'$ is always positive, and it takes its maximum, if $k' = 1$, $k = d_r$, $i = n_{d_r}$ and $i' = 1$, when it is $\omega - 2$, by (19). Thus

$$1 \leq n_{k'} + \cdots n_{k-1} + i - i' \leq n_1 + \cdots + n_{d_r} - 1 = \omega - 2$$

follows, implying that the quantity $((k - k')\omega - (n_{k'} + \cdots n_{k-1} + i - i'))$, is never 0 or 1 modulo ω.

To verify (R3*) let us note first that the sets, B_{uv} for $u \in \Gamma_v$, as defined in (23) are pairwise disjoint, and consecutive, i.e. form an interval of length

$$\sum_{u \in \Gamma_v} |B_{uv}| = \sum_{u \in \Gamma_v} (\omega - |T_{vu}|) = d_v\omega - |V' \setminus \{v\}| = (d_r - 1)\omega + 1,$$

for all $v \in C$, and hence the complementary set $V' \setminus \bigcup_{u \in \Gamma_v} B_{uv}$ has its cardinality as a multiple of ω (since $n = \alpha\omega + 1$). Thus it can be tiled by ω-cliques of the web G'. Therefore, to verify (R3*), we need to show first that the above hold with the definitions in (23), and second that to tile the sets $V' \setminus \bigcup_{u \in \Gamma_v} B_{uv}$ for $v \in C$ by ω-cliques of G' one does not need the cliques defined in (22).

To see the first part is easy just by looking at the definitions (23). For the feet there is nothing to check and for the head we have the sets

$$B_{[(k,n_k),(k,n_k+1)]} = [k\omega - (n_1 + \cdots n_{k-1}), (k+1)\omega - (n_1 + \cdots n_{k-1} + n_k)) \tag{24}$$

for $k = 1, 2, ..., d_r$, and these obviously are consecutive, in this order, with no overlap. For an interior vertex (k, i) of a leg (i.e. with $1 < i < n_k$) we have the two sets

$$\begin{aligned} B_{[(k,i+1),(k,i)]} &= [k\omega - (n_1 + \cdots n_{k-1} + i), k\omega - (n_1 + \cdots n_{k-1})) \text{ and} \\ B_{[(k,i-1),(k,i)]} &= [k\omega - (n_1 + \cdots n_{k-1}), (k+1)\omega - (n_1 + \cdots n_{k-1} + i - 1)) \end{aligned} \tag{25}$$

and again these sets are always consecutive without any overlap.

For the second part, let us first have a look again at the sets (24), and let us observe that the complement of their union can be partitioned by the cliques $\mathcal{H}_r = \{C'_{(d_r+j)\omega+1} | j = 0, 1, ..., \alpha - d_r\}$. Since for the cliques of the form $C'_{k\omega-(n_1+\cdots n_{k-1}+i)}$ for $1 \leq i \leq n_k$ for $1 \leq k \leq d_r$ (see (22)), we have

$$\omega - 1 \leq k\omega - (n_1 + \cdots n_{k-1} + i) \leq (d_r - 1)\omega + 1$$

therefore, $\mathcal{H}_r$ indeed does not contain any of these. For the two sets finally in (25), we can see that their complement is partitioned by the cliques

$$\mathcal{H}_{(k,i)} = \{C_{(k+j)\omega-(n_1+\cdots n_{k-1}+i-1)} | j = 1, ..., \alpha - 1\}$$

and again these are all different from those in (22). □

As an illustration, let us consider the $(2, 5)$-web (anti-hole) on 11 vertices, and the spider shown in Figure 1. In this example we have $\alpha = 2$, $\omega = 5$, (and hence $n = 11$), and, as shown in Figure 1, $r = (1, 2) = (2, 3) = (3, 2)$, $a = (1, 1)$, $b = (2, 2)$, $c = (2, 1)$, and $d = (3, 1)$. Then the sets by (22) and (23) are as follows

$$\begin{array}{lll} C'_{ar} = [4, 9) & B_{ra} = [4, 5) & B_{ar} = [5, 9) \\ C'_{br} = [7, 1) & B_{rb} = [7, 9) & B_{br} = [9, 1) \\ C'_{dr} = [0, 5) & B_{rd} = [0, 1) & B_{dr} = [1, 5) \\ C'_{bc} = [8, 2) & B_{bc} = [8, 9) & B_{cb} = [9, 2) \end{array}$$

The eight sets $[4, 5) \cup \{r, b, c, d\}$, $[5, 9) \cup \{a\}$, $[7, 9) \cup \{r, a, d\}$, $[9, 1) \cup \{b, c\}$, $[0, 1) \cup \{r, a, b, c\}$, $[1, 5) \cup \{d\}$, $[8, 9) \cup \{r, a, b, d\}$, and $[9, 2) \cup \{c\}$ together with $C = \{r, a, b, c, d\}$ and the seven of the original cliques of the $(2, 5)$-web, namely $[1, 6)$, $[2, 7)$, $[3, 8)$, $[5, 10)$, $[6, 0)$, $[9, 3)$ and $[10, 4)$ form the clique family of a $(3, 5)$-partitionable graph on the 16 vertices of $\mathbb{Z}_{11} \cup C$.

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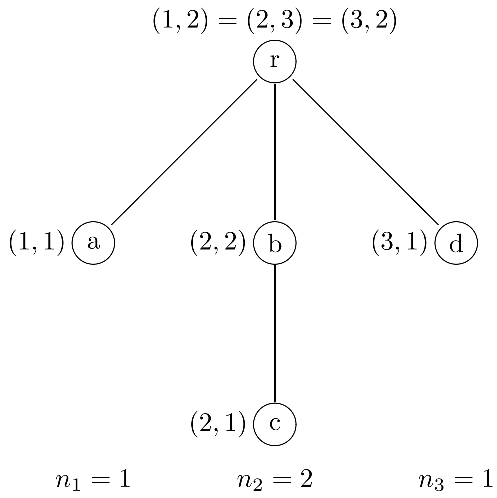

Figure 1: A coordinatized spider on 5 vertices.

Remark 6 *Even though for $\omega = 3$ every spider is a simple path of two edges, still, depending on where the head is, we get different results. For example, if we start from the $(2,3)$-web and the spider $\{r, a, b\}$ forms a 2 edge path with its head at the end then we obtain a $(3,3)$-web. While if the spider $\{a, r, b\}$ forms a 2 edge path again but its head is now in the middle then we get the other $(3,3)$-partitionable graph, which is not a web, see [3].*

Remark 7 *By the above result, we can generate an $(\alpha+1, \omega)$-partitionable graph from an (α, ω)-web for every labeled spider T on ω points with d_r legs, whenever $d_r \leq \alpha + 1$. Yet, some of these graphs might be isomorphic.*

Remark 8 *Obviously, $|L(T)| \geq d_v$ for every tree T and for every vertex $v \in T$, and there exists a vertex v in T such that $|L(T)| = d_v$ if and only if T is a spider.*

8 $(\alpha, 3)$-partitionable families and other experimental results.

For $\omega = 3$ we have the following characterization of critical cliques:

Lemma 8 *A clique is critical if and only if it is in the middle of a gem.*

Proof. There is a unique tree with 3 vertices, let us say b, c, d. There is a unique tree-covering family: $\{(b, c), (d), \ (b), (c, d)\}$. Thus there should exist cliques (a, b, c) and (c, d, e). Vertices a and e are different, otherwise we would get a K_4. Vertices a, b, c, d, e form a gem with critical clique (b, c, d) in the middle. $\square$

We conjecture that for $\omega = 3$ every partitionable graph has a critical clique. The following experimental results support this conjecture. We have verified, that for $\omega = 3$ there exists a gem (and therefore a critical clique) in all partitionable graphs with α up to 9; the existence of a diamond was verified for α up to 10.

In Table 1 we list some additional experimental results. We have generated all the partitionable graphs for $\omega = 3$ and $\alpha = 2, \ldots 7$ and for $\omega = 4$ and $\alpha = 4$ and 5. For $\omega = 3$ all graphs have critical cliques, while for $\omega = 4$ this is no longer true.

The column "ST" counts the number of graphs which have a *small transversal*, that is a subset of the vertices of size $\alpha + \omega - 1$ that intersects all ω-cliques and all α-stable sets. The column "C_5" lists the

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number of partitionable graphs without C_5. Both these values turn out to be useful parameters in case one is interested to generate partitionable graphs that are reasonable candidates to be counterexamples to the Strong Perfect Graph Conjecture. It is well-known that such a counterexample can not have neither a small transversal nor a C_5.

Table 1: The number of normalized partitionable graphs. (Numbers in bold were not known before)

				# of graphs without			# of graphs constructable by	
n	ω	α	# total	crit. clique	ST	C_5	CGPW	our construction
10	3	3	2	0	0	0	2	**2**
13	3	4	5	0	0	1	4	**5**
16	3	5	21	0	0	2	**18**	**21**
19	3	6	154	0	0	7	**138**	**154**
22	3	7	**1488**	**0**	**0**	**22**	**1332**	**1488**
17	4	4	132	6	1	1	**22**	**126**
21	4	5	**8340**	**1431**	0	**4**	**1189**	**6909**
25	4	6	?	?	**0**	?	?	?

Remark 9 *Our computations show that a counterexample to the Strong Perfect Graph Conjecture must have at least 26 vertices. This slightly improves the previous bound 25 given by Gurvich and Udalov (1992). These two bounds are obtained due to a computer analysis of the (4,6)- and (4,5)-graphs, respectively. It was shown that all these graphs have small transversals and thus cannot be counterexamples to the Berge Conjecture. To reach the next bound 29 the case of (5,5)-graphs has to be considered. (Let us note that no counterexample can exist among* $(\alpha, 3)$*-graphs, as it was shown in [12].)*

References

[1] C. Berge. Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind (Zusammenfassung), Wiss. Z. Martin-Luther-Univ., Halle-Wittenberg, Math.-Natur. Reihe (1961) 114–115.

[2] R.G. Bland, H.C. Huang and L.E. Trotter Jr. Graphical properties related to minimal imperfection, *Discrete Mathematics* **27** (1979) pp. 11-22.

[3] V. Chvátal, R.L. Graham, A.F. Perold and S.H. Whitesides. Combinatorial designs related to the perfect graph conjecture, *Discrete Mathematics* **26** (1979) pp. 83-92.

[4] V. Gurvich and V. Udalov. Berge Strong Perfect Graph Conjecture holds for the graphs with less than 25 vertices. *Manuscript* (1992), 23 p.

[5] L. Lovász. A characterization of perfect graphs, *J. Combinatorial Theory*, Ser.B **13** (1972) pp. 95-98.

[6] E. Olaru. Zur Charakterisierung perfekter Graphen. *Elektronische Informationsverarbeitung und Kybernetik (EIK)* **9** (1973) pp. 543-548.

[7] M.W. Padberg. Perfect zero-one matrices, *Math. Programming* **6** (1974) pp. 180-196.

[8] B. Reed. *A semi strong perfect graph theorem.* Ph.D. Thesis, McGill University, Québec (1986).

[9] A. Sebő. Forcing colorations and the perfect graph conjecture. In: *Integer Programming and Combinatorial Optimization 2* (E. Balas and R. Kannan, eds.) Mathematical Programming Society and Carnegie Mellon University, 1992.

This paper appeared in: Journal of Graph Theory 41 (2002), 259-285

[10] A. Sebő. The connectivity of minimal imperfect graphs. *Journal of Graph Theory* **23**(1) (1996a) pp. 77-85.

[11] A. Sebő. On critical edges in minimal imperfect graphs. *J. Combinatorial Theory*, Ser.B **67** (1996b) pp. 62-85.

[12] A. Tucker. The validity of the perfect graph conjecture for K_4-free graphs. In: *Annals of Discrete Mathematics, Topics on Perfect Graphs* (Berge and Chvátal, eds.), **21** (1984) pp. 149-157.