Exercises 8

1. Prove Proposition 4.2: Let \( G \) be a digraph, \( c : E(G) \to \mathbb{R} \), and \( F \subseteq E(G) \) such that \( (V(G), E(G) \setminus F, c) \) is conservative. Then

\[
\max \{ \min \{ c_\pi(e) : e \in F \} : \pi : V(G) \to \mathbb{R}, c_\pi(e) \geq 0 \text{ for } e \in E(G) \setminus F \} = \min \{ \frac{c(E(C))}{|F \cap E(C)|} : C \text{ circuit in } G, F \cap E(C) \neq \emptyset \} = \max \{ \lambda : c_\lambda \text{ is conservative} \}.
\]

[5 points]

2. Consider the linear Time-Cost Tradeoff Problem.

(a) Show that the deadline version is the dual of a Minimum Cost Flow Problem. [3 points]

(b) Use (a) to develop a combinatorial polynomial time algorithm for the budget version. [2 points]

3. Prove Proposition 5.1: Let \( (G, ([a_e, b_e])_{e \in E(G)}, c) \) be an instance of the linear Time-Cost Tradeoff Problem. Let \( x \) be a time-cost optimal solution and let \( x' \) be a solution with \( T(x') < T(x) \). Describe a feasible solution of the following linear program:

\[
\begin{align*}
\min \quad & \sum_{e \in E_x} \gamma_e y_e \\
\text{s.t.} \quad & y_e \geq 0 \quad (e \in E^b_x) \\
& y_e \leq 0 \quad (e \in E^a_x) \\
& \zeta_w \leq \zeta_v + y_e \quad (e = (v, w) \in E_x) \\
& \zeta_t - \zeta_s = 1,
\end{align*}
\]

where \( E_x := \bigcup \{ E(P) : P \in \mathcal{P}(G), \sum_{e \in E(P)} x_e = T(x) \} \) is the set of critical edges, and \( E_x^a := \{ e \in E_x : x_e = a_e \} \) and \( E_x^b := \{ e \in E_x : x_e = b_e \} \). [5 points]

The deadline for submitting solutions is June 23 before the lecture (12:15).