Exercise 14.1:

1. Prove: A greedoid is a matroid if and only if it is an interval greedoid without lower bounds. (2 Points)

2. Prove: A greedoid is an antimatroid if and only if it is an interval greedoid without upper bounds. (2 Points)

Exercise 14.2:

Let \(X = \{x_1, \ldots, x_m\}\) be a finite set of discrete random variables, each taking values in a finite set \(Y\) and having probability distribution \(p\). For \(S \subseteq \{1, \ldots, m\}\) we define

\[H(S) = - \sum_{z \in Y^{\lfloor |S| \rfloor}} p(x_s = z, \forall s \in S) \cdot \log(p(x_s = z, \forall s \in S))\]

to be the Shannon-Entropy. Prove that \(H\) is submodular. (4 Points)

Exercise 14.3:

Let \(G = (V, E)\) be an undirected graph. For a set \(X \subseteq V(G)\) let \(f(X)\) denote the number of edges in \(E\) incident to \(X\).

1. Prove that \(f\) is a submodular function. (2 Points)

2. Prove that it is NP-hard to find a set \(X \subseteq V\) maximizing \(f(X)\). (2 Points)

Let \(y : V(G) \to \mathbb{N}\) be a function. We want to find an orientation of \(G\) (i.e. a directed graph \(G'\) such that \(V(G) = V(G')\) and the underlying undirected graph is equal to \(G\)) such that

\[|\delta^-(v)| = y(v)\] for each \(v \in V(G)\). (1)

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3. Show that such an orientation exists if and only if

\[ y(V) = |E| \quad \text{and} \quad y(X) \leq f(X) \forall X \subseteq V(G). \]  \hspace{1cm} (2)

*Hint:* Construct a network \((V', E', u)\) with \(V' = E \cup V \cup \{s, t\}\), \(E' = \{(s, e) \mid e \in E\} \cup \{(v, t) \mid v \in V\} \cup \{(e, v) \mid v \in e\}\) and a suitable capacity function \(u\) such that a flow \(g\) in \((V', E', u)\) corresponds to an orientation of \(\sum_{e \in \delta^+(s)} g(e)\) edges in \(E(G)\). Use the MAX-FLOW-MIN-CUT Theorem.

(2 Points)

4. Give a polynomial time combinatorial algorithm which either finds

- an orientation as desired or
- a set \(X \subseteq V(G)\) which serves as certificate that such an orientation does not exist.

(1 Point)

5. Consider the following alternative algorithm:

If there is no orientation as desired, stop. Otherwise, start with \(G' = (V(G), \emptyset)\). For each edge \(e = \{v, w\} \in E(G)\), set \(G' := G' + (v, w)\), \(G := G - e\) and \(y(w) := y(w) - 1\) if there exists an orientation satisfying \([\square]\) for \(G - e\) after decreasing \(y(w)\) by 1. If this is not the case, set \(G' := G' + (w, v)\), \(G := G - e\), \(y(v) := y(v) + 1\).

The question, if an orientation satisfying \([\square]\) exists can be answered by using Schrijver’s algorithm to find a set \(X\) minimizing the submodular function \(f(X) - y(X)\).

Compare the running-time of this algorithm to the running-time of your algorithm from exercise 14.3.4.

(1 Point)

**Deadline:** Tuesday, January 29, 2013, before the lecture.