Combinatorial Optimization

Exercise Sheet 10

Exercise 10.1:
Let $M = (E, F)$ be a matroid, $B \subseteq E$, and $J \subset E$ a basis of $B$. We define $M/B := (E \setminus B, \{J' \subseteq E \setminus B \mid J' \cup J \in F\})$. Prove:

1. $M/B$ is a matroid that does not depend on the choice of $J$. (2 Points)

2. The rank function of $M/B$ is given by $r'(A) = r(A \cup B) - r(B)$ for all $A \subseteq E \setminus B$. (2 Points)

3. Let $\emptyset = T_0 \subseteq T_1 \subseteq \ldots \subseteq T_{l+1} = E$. The bases of $T_i$ in $M$ that intersect $T_i$ in a basis of $T_i$ for each $i \in \{1, \ldots, l\}$ are the bases of $T_i$ in the matroid $N := N_0 \oplus N_1 \oplus \ldots \oplus N_l$, where $N_i := (M/T_i) \setminus T_{i+1}$. (4 Points)

Exercise 10.2:
Let $M = (E, F)$ be a matroid.

1. Let $X \subseteq E$. Prove: Let $Y_1$ be a base of $M \setminus X$ and $Y_2$ be a base of $M/(E \setminus X)$, then $Y_1 \cup Y_2$ is a base of $M$. (2 Points)

Now let $N$ and $K$ be nonempty subsets of $E$. A game $\langle M; N, K \rangle$ is played as follows: Angelika and Bodo (who plays first) alternatingly tag different elements of $N$. A tagged element cannot be tagged again later in the game. Angelika wins if she tags a set of elements that span $K$. Bodo wins if all elements of $N$ are tagged and Angelika did not win.

2. Prove: If $N$ contains two disjoint subsets $A_0$ and $B_0$ which span each other and which both span $K$, then Angelika can win against any strategy Bodo might have. Hint: Assume Bodo picks $a_0 \in A_0$, then there is a $b_0 \in B_0$ such that $(A_0 \setminus \{a_0\}) \cup \{b_0\}$ is a base of $M_0 := M \setminus \sigma(A_0 \cup B_0)$. Also, Exercise 10.1 might be helpful. (4 Points)

Note: The other direction is also true, but harder to prove. You may use this fact for the next exercise.

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Now Angelika and Bodo play the even funnier game \(\langle G; u, v \rangle\). Here \(G\) is a graph and \(u\) and \(v\) are vertices of \(G\). Angelika and Bodo alternatingly tag edges. Angelika wins if her edges contain a \(u\)-\(v\)-path and Bodo wins if Angelika didn’t win when all edges are tagged. Again Bodo plays first.

3. Prove: Angelika has a winning strategy if and only if there are \(V' \subseteq V(G)\), \(E_1 \subseteq E(G)\), and \(E_2 \subseteq E(G)\) with \(\{u, v\} \subseteq V'\) and \(E_1 \cap E_2 = \emptyset\) such that \((V', E_1)\) and \((V', E_2)\) are trees. (2 Points)

**Deadline:** Tuesday, December 18, 2012, before the lecture.