Exercise 6.1:

We call a directed graph $G = (V,E)$ special if the underlying undirected graph is bipartite, loopless, and has no parallel edges, and there exists a permutation $\pi : V \to V$ such that $\pi(v) \neq v$ and $\pi(\pi(v)) = v$ hold for all $v \in V$ and $(v,w) \in E$ implies $(\pi(w),\pi(v)) \in E$. An edge labeling $x : E \to \mathbb{N}_+$ on a special graph is called balanced if $x(v,w) = x(\pi(w),\pi(v))$ for every $(v,w) \in E$. If $G$ is a special graph, $u : E \to \mathbb{N}_+$ are edge capacities, and $f$ is a $b$-flow in $(G,u)$, then a walk $P$ in $G$ is called valid if it does not contain both $(v,w)$ and $(\pi(w),\pi(v))$ for some $(v,w) \in E$ such that $u_f(v,w) = u_f(\pi(w),\pi(v)) = 1$.

(i) Prove: Let $G = (V,E)$ be a special graph with balanced edge capacities $u : E \to \mathbb{N}_+$ and $f$ and $g$ different balanced circulations on $(G,u)$. Then there are valid cycles $C_1, \ldots, C_k$ in the residual graph $G_f$ and functions $f_{C_i} : E(C_i) \to \{-1,0,1\}$ for $i \in \{1,\ldots,k\}$ such that

$$g(v,w) - f(v,w) = \sum_{i=1}^{k} (f_{C_i}(v,w) + f_{C_i}(\pi(w),\pi(v))) \quad \text{for all } (v,w) \in E.$$

*Hint:* Remember the proof of the decomposition theorem for ordinary flows.

Now let $(G,u,s,t)$ be a network with balanced edge capacities $u$ and $f$ a balanced $s$-$t$-flow in this network.

(ii) Show that $f$ is a balanced $s$-$t$-flow with maximum value if and only if there is no valid $s$-$t$-path in $G_f$.

(iii) Assume that it is possible in polynomial time to find a valid $s$-$t$-path in $G_f$ or to decide that none exists. Describe an algorithm that finds a maximum balanced flow in polynomial time.

We want to solve the cardinality matching problem in a simple graph $G = (V,E)$. From this, we construct a directed graph as follows. Let $V' := \{v' \mid v \in V\}$ be a copy of $V$ and $\{s,t\}$ two additional vertices. Furthermore, let $E_G := \{(v,w') \mid \{v,w\} \in E\}$, $E_s := \{(s,v) \mid v \in V\}$, and $E_t := \{(v',t) \mid v \in V\}$, and define the directed graph $D_G := (V \cup V' \cup \{s,t\},E_G \cup E_s \cup E_t)$. (Continued on next page.)
(iv) Show how the algorithm from (iii) and \( D_G \) can be used to find maximum matchings in polynomial time.

\[(4+2+2+4 \text{ Points})\]

**Exercise 6.2:**
Let \( G \) be a graph and \( P \) the fractional perfect matching polytope of \( G \). Prove that the vertices of \( P \) are exactly the vectors \( x \) with

\[
x_e = \begin{cases} 
\frac{1}{2} & \text{if } e \in E(C_1) \cup \cdots \cup E(C_k) \\
1 & \text{if } e \in M \\
0 & \text{otherwise,}
\end{cases}
\]

where \( C_1, \ldots, C_k \) are vertex disjoint odd circuits and \( M \) is a perfect matching in \( G - (V(C_1) \cup \cdots \cup V(C_k)) \).

\[(4 \text{ Points})\]

**Exercise 6.3:**
Let \( G \) be a graph, \( T \subseteq V(G) \) with \(|T|\) even, and \( F \subseteq E(G) \). A subset \( C \subseteq E(G) \) is called a \( T \)-cut if \( C = \delta(U) \) for some \( U \subseteq V(G) \) with \(|U \cap T|\) odd. Prove:

(i) \( F \) has nonempty intersection with every \( T \)-join if and only if \( F \) contains a \( T \)-cut.

(ii) \( F \) has nonempty intersection with every \( T \)-cut if and only if \( F \) contains a \( T \)-join.

\[(2+2 \text{ Points})\]

**Deadline:** Tuesday, November 20, 2012, before the lecture.