Professor Dr. Stephan Held
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Exercise Set 12

Exercise 12.1. Let U be a finite set and $f\colon 2^U \to \mathbb{R}$. Prove that f is submodular if and only if $f(X \cup \{y, z\}) - f(X \cup \{y\}) \leq f(X \cup \{z\}) - f(X)$ for all $X \subseteq U$ and $y, z \in U$ with $y \neq z$.

(3 points)

Exercise 12.2. Let (G, u, s, t) be a network and $U := \delta^+(s)$. Let

$$P := \Big\{x \in \mathbb{R}^U_+ : \text{there is an } s\text{-}t \text{ flow } f \text{ in } (G, u) \text{ with } f(e) = x_e \text{ for all } e \in U\Big\}.$$

Prove that P is a polymatroid.

(4 points)

Exercise 12.3. Let $f\colon 2^U \to \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$. Prove that the set of vertices of the base polyhedron of f is precisely the set of vectors $b^{\prec}$ for all total orders $\prec$ of U, where

$$b^{\prec}(u) := f\Big(\{v \in U : v \preceq u\}\Big) - f\Big(\{v \in U : v \prec u\}\Big) \qquad (u \in U).$$

(5 points)

Exercise 12.4. Let $f\colon 2^U \to \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$, and let $B(f)$ denote its base polyhedron. Prove that

$$\min\{f(X) : X \subseteq U\} = \max\Big\{x^-(U) : x \in B(f)\Big\}$$

where $x^-(U) = \sum_{u \in U} \min\{0, x(u)\}$.

(4 points)

Deadline: January 19, before the lecture. The websites for lecture and exercises can be found at:

`https://ecampus.uni-bonn.de/goto_ecampus_crs_2772883.html`

In case of any questions feel free to contact me at armbruster@or.uni-bonn.de.