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Exercise Set 9

Exercise 9.1. Let G be a graph, $u\colon E(G) \to \mathbb{N} \cup \{\infty\}$ and $b\colon V(G) \to \mathbb{N}$. Prove Theorem 2.29 from the lecture, i.e. show that (G, u) has a perfect b-matching if and only if for any two disjoint subsets $X, Y \subseteq V(G)$ the number of connected components C in $G - X - Y$ for which $\sum_{c \in V(C)} b(c) + \sum_{e \in E_G(v(C),Y)} u(e)$ is odd does not exceed

$$\sum_{v \in X} b(v) + \sum_{y \in Y} \left(\sum_{e \in \delta(y)} u(e) - b(y) \right) - \sum_{e \in E_G(X,Y)} u(e)$$

(6 points)

Exercise 9.2. Let G be an undirected graph and $T \subseteq V(G)$ with $|T| = 2k$ even. Prove that the minimum cardinality of a T-cut in G equals the maximum of $\min_{i=1}^{k} \lambda_{s_i,t_i}$ over all pairings $T = \{s_1, t_1, \ldots, s_k, t_k\}$, where $\lambda_{s,t}$ denotes the maximum number of pairwise edge-disjoint s-t-paths.

(4 points)

Exercise 9.3. Find an instance (G, T, c) of the minimum weight T-join problem such that c is conservative and there exists a minimum weight T-join J that can not be written as a disjoint union of the edge sets of $\frac{|T|}{2}$ c-shortest paths with endpoints in T and possibly some zero-weight circuits.

(2 points)

Exercise 9.4. Let G be an undirected graph and $T \subseteq V(G)$ such that $|T|$ is even. Show that the convex hull of incidence vectors of T-joins in G is given by the set P of all $x \in [0,1]^{E(G)}$ satisfying

$$\sum_{e \in F} (1 - x_e) + \sum_{e \in \delta(X) \setminus F} x_e \geq 1 \forall X \subseteq V(G),\ F \subseteq \delta(X) \text{ such that } |X \cap T| + |F| \text{ is odd.}$$

Hint: First show that the incidence vector of every T-join in G satisfies the given constraints. Next, show that every vertex of P is the incidence vector of a T-join in G by showing that for each cost function $c \in \mathbb{R}^{E(G)}$, there exists a T-join J such that the incidence vector χ^J of J minimizes $c^T x$ over P. To this end, let

$E^- := \{e \in E(G) : c(e) < 0\}$ and, for a vector $x \in P$, consider the vector $\bar{x}$ given by

$$\bar{x}_e = \begin{cases} x_e & , e \in E(G) \setminus E^- \\ 1 - x_e & , e \in E^- \end{cases}$$

and prove that $\bar{x}$ is contained in the up-hull of the $T\Delta\text{odd}(E^-)$-polyhedron of G. Finally, recall Lemma 2.30 and its proof.

(4 points)

Deadline: December 15, before the lecture. The websites for lecture and exercises can be found at:

`https://ecampus.uni-bonn.de/goto_ecampus_crs_2772883.html`

In case of any questions feel free to contact me at armbruster@or.uni-bonn.de.