Professor Dr. Stephan Held
Susanne Armbruster, M. Sc.

Exercise Set 2

Exercise 2.1. Let G be a bipartite graph. For each $v \in V(G)$, let $<_v$ be a linear ordering of $\delta(v)$. Prove that there is a matching $M \subseteq E(G)$ such that for each $e \in E(G) \setminus M$ there is an edge $f \in M$ and a vertex $v \in V(G)$ such that $v \in (e \cap f)$ and $e <_v f$.

(5 points)

Exercise 2.2. Let G be a graph, $n := |V(G)|$ even, and for any set $X \subseteq V(G)$ with $|X| \leq \frac{3}{4}n$ we have

$$\left| \bigcup_{x \in X} \Gamma(x) \right| \geq \frac{4}{3}|X|.$$

Prove that G has a perfect matching.

Hint: Let S be a set violating the Tutte condition. Prove that the number of connected components in $G - S$ with just one element is at most $\max\left\{0, \frac{4}{3}|S| - \frac{1}{3}n\right\}$. Consider the cases $|S| \geq \frac{n}{4}$ and $|S| < \frac{n}{4}$ separately.

(5 points)

Exercise 2.3. Let $S = \{1, \dots, n\}$ for some $n \geq 1$.

(i) Suppose $0 \leq k \leq n - 1$ and consider the bipartite graph $G = (A \dot{\cup} B, E)$ where

$$\begin{aligned} A &:= \{X \subseteq S \,:\, |X| = k\}, \\ B &:= \{Y \subseteq S \,:\, |Y| = k + 1\}, \\ E &:= \{\{X, Y\} \,:\, X \in A,\, Y \in B,\, X \subseteq Y\}. \end{aligned}$$

Show that there is a matching covering A if $k < n/2$, and that there is a matching covering B if $k > n/2 - 1$.

(ii) Suppose $\mathcal{F}$ is a family of subsets of S with the property that no element of $\mathcal{F}$ is contained in another element of $\mathcal{F}$. Show that:

$$|\mathcal{F}| \leq \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}$$

and that this bound is tight (for every n).

(2+4 points)

Professor Dr. Stephan Held
Susanne Armbruster, M. Sc.

Deadline: October 27. Further information on lecture and exercises can be found in the corresponding eCampus course at:

https://ecampus.uni-bonn.de/goto_ecampus_crs_2772883.html

In case of any questions feel free to contact me at armbruster@or.uni-bonn.de.