Exercise Set 12

Exercise 12.1. Let $U$ be a finite set and $f : 2^U \to \mathbb{R}$. Prove that $f$ is submodular if and only if $f(X \cup \{y, z\}) - f(X \cup \{y\}) \leq f(X \cup \{z\}) - f(X)$ for all $X \subseteq U$ and $y, z \in U$ with $y \neq z$. 

(5 points)

Exercise 12.2. Let $0 < \epsilon < \frac{1}{2}$ be fixed and $n \in \mathbb{N}$ even with $\epsilon n \in \mathbb{N}$. Let $U = \{1, \ldots, n\}$. For any $C \subseteq U$ with $2|C| = |U|$ consider the functions $g, f_C : 2^U \to \mathbb{Z}_+$ defined as follows: For $S \subseteq U$ let $k := |S \cap C|$ and $l := |S \setminus C|$, and let $g(S) := |S||U\setminus S|$ and $f_C(S) := g(S)$ if $|k - l| \leq \epsilon n$ and $f_C(S) := n|S| - 4kl + \epsilon^2 n^2 - 2\epsilon n|k - l|$ if $|k - l| \geq \epsilon n$.

(i) Show that the two definitions of $f_C(S)$ coincide if $|k - l| = \epsilon n$.

(ii) Show that $g$ and $f_C$ are submodular. Hint: Use Exercise 12.1.

(iii) Observe that an algorithm is likely to need exponentially many oracle calls to find out which of these functions ($g$ or $f_C$ for some $C$) is the input.

(iv) Show that the maximum values of $g$ and any $f_C$ differ by a factor more than $2(1 - 2\epsilon)$.

(3 + 3 + Bonus* + 4 points)

* Bonus points given for [iii] make up for points missing in [i], [ii] and [iv].

Exercise 12.3. Let $S$ be a finite set and let $b_1, b_2 : 2^S \to \mathbb{R}$ be two submodular functions. Furthermore, let $S'$ and $S''$ be two disjoint copies of $S$. Set $V = S' \cup S''$ and

$$C = \{U' : U \subseteq S\} \cup \{S' \cup U'' : U \subseteq S\},$$

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where $U'$ and $U''$ denote the two copies of $U \subseteq S$ in $S'$ and $S''$, and define $b : C \to \mathbb{R}_{\geq 0}$ by

\[
\begin{align*}
    b(U') & := b_1(U) & \text{for } U \subseteq S, \\
    b(V \setminus U'') & := b_2(U) & \text{for } U \subseteq S, \\
    b(S') & := \min\{b_1(S), b_2(S)\}.
\end{align*}
\]  

(i) Show that $C$ is a crossing family.

(ii) Show that $b$ is crossing submodular on $C$.

(2+3 points)

**Deadline:** January 16, before the lecture.