Exercises Set 11

Exercise 11.1. Prove that a nonempty compact set \( P \subseteq \mathbb{R}_+^n \) is a polymatroid if and only if

(i) For all \( 0 \leq x \leq y \in P \) we have \( x \in P \).

(ii) For all \( x \in \mathbb{R}_+^n \) and all \( y, z \leq x \) with \( y, z \in P \) that are maximal with this property (i.e. \( y \leq w \leq x \) and \( w \in P \) implies \( w = y \), and \( z \leq w \leq x \) and \( w \in P \) implies \( w = z \)) we have \( 1_y = 1_z \), where \( 1 \) is the vector whose entries are all 1.

(4 points)

Exercise 11.2. Let \((G, u, s, t)\) be a network and \( U := \delta^+(s) \). Let

\( P := \{ x \in \mathbb{R}_+^U : \text{there is an } s-t \text{ flow } f \text{ in } (G, u) \text{ with } f(e) = x_e \text{ for all } e \in U \} \).

Prove that \( P \) is a polymatroid.

(4 points)

Exercise 11.3. Let \( f : 2^U \to \mathbb{R} \) be a submodular function with \( f(\emptyset) = 0 \). Prove that the set of vertices of the base polyhedron of \( f \) is precisely the set of vectors \( b^\prec \) for all total orders \( \prec \) of \( U \), where

\( b^\prec(u) := f\left(\{v \in U : v \preceq u\}\right) - f\left(\{v \in U : v \prec u\}\right) \quad (u \in U) \).

(6 points)

Exercise 11.4. Let \( f : 2^U \to \mathbb{R} \) be a submodular function with \( f(\emptyset) = 0 \), and let \( B(f) \) denote its base polyhedron. Prove that

\[
\min\{f(X) : X \subseteq U\} = \max\left\{ \sum_{u \in A} z_u : z \in \mathbb{R}_+^U \text{ with } \sum_{u \in A} z_u \leq \min\{0, f(A)\} \text{ for all } A \subseteq U \right\} \\
= \max\left\{ \sum_{u \in U} \min\{0, y_u\} : y \in B(f) \right\}.
\]

(6 points)
**Deadline:** January 9th, before the lecture. The websites for lecture and exercises can be found at:

[http://www.or.uni-bonn.de/lectures/ws19/co_exercises/exercises.html](http://www.or.uni-bonn.de/lectures/ws19/co_exercises/exercises.html)

In case of any questions feel free to contact me at [rabenstein@or.uni-bonn.de](mailto:rabenstein@or.uni-bonn.de).