Exercise Set 2

Exercise 2.1. Let $G$ be a graph and $M$ a matching in $G$ that is not maximum. In this exercise we use the terminology **disjoint subgraphs/paths/circuits** and mean it quite literally: Two subgraphs are *disjoint* if they have no edges and no vertices in common. (Note that the term *vertex-disjoint paths* is often used to mean that two paths have no *inner*-vertices in common, but possibly endpoints.)

(i) Show that there are $\nu(G) - |M|$ disjoint $M$-augmenting paths in $G$.

(ii) Show the existence of an $M$-augmenting path of length at most $\frac{\nu(G) + |M|}{\nu(G) - |M|}$.

(iii) Let $P$ be a shortest $M$-augmenting path in $G$ and $P'$ an $(M \Delta E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + 2 \cdot |E(P) \cap E(P')|$.

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then $P_i$ and $P_j$ are disjoint.

(v) Show that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains less than $2\sqrt{\nu(G)} + 1$ different numbers.

From now on, let $G$ be bipartite and set $n := |V(G)|$ and $m := |E(G)|$.

(vi) Given a non-maximum matching $M$ in $G$ show that we can find in $O(n + m)$-time a family $P$ of disjoint shortest $M$-augmenting paths such that if $M'$ is the matching obtained by augmenting $M$ over every path in $P$, then

$$\min\{|E(P)| : P \text{ is an } M'-\text{augmenting path}\}$$

$$> \min\{|E(P)| : P \text{ is an } M-\text{augmenting path}\}$$

(vii) Describe an algorithm with runtime $O(\sqrt{n}(m + n))$ that solves the **Cardinality Matching Problem** in bipartite graphs.

(1+1+2+2+2+3+1=12 points)
Exercise 2.2. Let $G$ be a 2-edge-connected graph, and let $\varphi(G)$ be the minimum number of even ears in any ear-decomposition of $G$. Show that then for every $v \in V(G)$ there is a matching in $G - v$ of cardinality $\frac{1}{2}(n - 1 - \varphi(G))$.

(4 points)

Exercise 2.3. The permanent of a square matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ is defined by

$$\text{per}(M) = \sum_{\pi \in S_n} \prod_{i=1}^{n} m_{i,\pi(i)}$$

where $S_n$ denotes the group of permutations of $\{1, \ldots, n\}$ by $S_n$. In this exercise, you may use the following results about the permanent of $M$.

- If all entries of $M$ are either 0 or 1 and its row sums are $r_1, \ldots, r_n$, then $\text{per}(M) \leq (r_1!)^{\frac{1}{n}} \cdots (r_n!)^{\frac{1}{n}}$. This was shown by Brègman[1973].

- If $M$ is a non negative $n \times n$ matrix whose collum and row sums are all equal to 1, then $\text{per}(M) \geq n! \left(\frac{1}{n}\right)^n$. This was conjectured by van der Waerden and later shown to be true by Falikman[1981] and Egoryčev[1980]. Such matrices are called doubly stochastic matrices.

Let $G$ be a balanced bipartite graph on $2n$ vertices, i.e. there is a bipartition $V(G) = A \cup B$ of $G$ with $|A| = |B| = n$. Recall $M_G(x)$ was defined in Exercise 1.4. Finally let $\Phi(G)$ denote the number of perfect matchings in $G$.

(a) Prove $\Phi(G)$ and $\text{per}(M_G(1))$ to be equal.

(b) In the case of a $k$-regular $G$, prove $n! \left(\frac{k}{n}\right)^n \leq \Phi(G) \leq (k!)^{\frac{n}{2}}$.

(2 + 4 points)

Deadline: October 24th, before the lecture. The websites for lecture and exercises can be found at:

http://www.or.uni-bonn.de/lectures/ws19/co_exercises/exercises.html

In case of any questions feel free to contact me at rabenstein@or.uni-bonn.de.