Professor Dr. Jens Vygen
Rudolf Scheifele

# Exercise Set 12

**Exercise 12.1.** Let $f\colon 2^U \to \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$, and let $B(f)$ denote its base polyhedron. Prove that

$$\begin{aligned}
&\min\{f(X) : X \subseteq U\}\\
&= \max\left\{\sum_{u\in U} z_u : z \in \mathbb{R}^U \text{ with} \sum_{u\in A} z_u \le \min\{0, f(A)\} \text{ for all } A \subseteq U\right\}\\
&= \max\left\{\sum_{u\in U} \min\{0, y_u\} : y \in B(f)\right\}.
\end{aligned}$$

(5 points)

**Exercise 12.2.** Consider the Simple Submodular Function Maximization Algorithm where the randomized step is replaced by setting $A := A \cup \{i\}$ if $\Delta_A \ge \Delta_B$ and $B := B \setminus \{i\}$ otherwise. Show that this algorithm is a 3-approximation algorithm.

(5 points)

**Exercise 12.3.** Let $0 < \epsilon < \frac{1}{2}$ be fixed and $n \in \mathbb{N}$ even with $\epsilon n \in \mathbb{N}$. Let $U = \{1, \dots, n\}$. For any $C \subset U$ with $2|C| = |U|$ consider the functions $g, f_C\colon 2^U \to \mathbb{Z}_+$ defined as follows: For $S \subseteq U$ let $k := |S \cap C|$ and $l := |S \setminus C|$, and let $g(S) := |S||U\setminus S|$ and $f_C(S) := g(S)$ if $|k-l| \le \epsilon n$ and $f_C(S) := n|S| - 4kl + \epsilon^2 n^2 - 2\epsilon n|k-l|$ if $|k-l| \ge \epsilon n$.

(i) Show that the two definitions of $f_C(S)$ coincide if $|k-l| = \epsilon n$.

(ii) Show that $g$ and $f_C$ are submodular. *Hint:* Use Exercise 10.4.

(iii) Observe that an algorithm is likely to need exponentially many oracle calls to find out which of these functions ($g$ or $f_C$ for some $C$) is the input.

(iv) Show that the maximum values of $g$ and any $f_C$ differ by a factor more than $2(1-2\epsilon)$.

(3 + 3 + Bonus* + 4 points)

* Bonus points given for (iii) make up for points missing in (i), (ii) and (iv).

Professor Dr. Jens Vygen
Rudolf Scheifele

**Information:** Submissions in groups of up to two students are allowed.

**Deadline:** Tuesday, January 22, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws18/coex.html`

In case of any questions feel free to contact me at scheifele@or.uni-bonn.de.