Exercise Set 12

Exercise 12.1. Let \( f : 2^U \to \mathbb{R} \) be a submodular function with \( f(\emptyset) = 0 \), and let \( B(f) \) denote its base polyhedron. Prove that

\[
\min \{ f(X) : X \subseteq U \} = \max \left\{ \sum_{u \in U} z_u : z \in \mathbb{R}^U \text{ with } \sum_{u \in A} z_u \leq \min \{0, f(A)\} \text{ for all } A \subseteq U \right\} = \max \left\{ \sum_{u \in U} \min \{0, y_u\} : y \in B(f) \right\}.
\]

(5 points)

Exercise 12.2. Consider the Simple Submodular Function Maximization Algorithm where the randomized step is replaced by setting \( A := A \cup \{i\} \) if \( \Delta_A \geq \Delta_B \) and \( B := B \setminus \{i\} \) otherwise. Show that this algorithm is a 3-approximation algorithm.

(5 points)

Exercise 12.3. Let \( 0 < \epsilon < \frac{1}{2} \) be fixed and \( n \in \mathbb{N} \) even with \( \epsilon n \in \mathbb{N} \). Let \( U = \{1, \ldots, n\} \). For any \( C \subseteq U \) with \( 2|C| = |U| \) consider the functions \( g, f_C : 2^U \to \mathbb{Z}_+ \) defined as follows: For \( S \subseteq U \) let \( k := |S \cap C| \) and \( l := |S \setminus C| \), and let \( g(S) := |S||U \setminus S| \) and \( f_C(S) := g(S) \) if \( |k - l| \leq \epsilon n \) and \( f_C(S) := n|S| - 4kl + \epsilon^2 n^2 - 2\epsilon n|k - l| \) if \( |k - l| \geq \epsilon n \).

(i) Show that the two definitions of \( f_C(S) \) coincide if \( |k - l| = \epsilon n \).

(ii) Show that \( g \) and \( f_C \) are submodular. \textit{Hint:} Use Exercise 10.4.

(iii) Observe that an algorithm is likely to need exponentially many oracle calls to find out which of these functions (\( g \) or \( f_C \) for some \( C \)) is the input.

(iv) Show that the maximum values of \( g \) and any \( f_C \) differ by a factor more than \( 2(1 - 2\epsilon) \).

(3 + 3 + Bonus* + 4 points)

* Bonus points given for (iii) make up for points missing in (i), (ii) and (iv).
Information: Submissions in groups of up to two students are allowed.

Deadline: Tuesday, January 22, before the lecture. The websites for lecture and exercises can be found at:

http://www.or.uni-bonn.de/lectures/ws18/coex.html

In case of any questions feel free to contact me at scheifele@or.uni-bonn.de.