Exercise Set 11

Exercise 11.1. Let (G, u, s, t) be a network and $U := \delta^+(s)$. Let

$$P := \Big\{x \in \mathbb{R}_+^U : \text{there is an } s\text{-}t \text{ flow } f \text{ in } (G,u) \text{ with } f(e) = x_e \text{ for all } e \in U\Big\}.$$

Prove that P is a polymatroid.

(5 points)

Exercise 11.2. Prove that a nonempty compact set $P \subseteq \mathbb{R}_+^n$ is a polymatroid if and only if

(i) For all $0 \le x \le y \in P$ we have $x \in P$.

(ii) For all $x \in \mathbb{R}_+^n$ and all $y, z \le x$ with $y, z \in P$ that are maximal with this property (i.e. $y \le w \le x$ and $w \in P$ implies $w = y$, and $z \le w \le x$ and $w \in P$ implies $w = z$) we have $\mathbb{1}y = \mathbb{1}z$, where $\mathbb{1}$ is the vector whose entries are all 1.

(5 points)

Exercise 11.3. Let S be a finite set and $f\colon 2^S \to \mathbb{R}$ with $f(\emptyset) = 0$. Define $f'\colon \mathbb{R}_+^S \to \mathbb{R}$ as follows: For any $x \in \mathbb{R}_+^S$ there are unique $k \in \mathbb{Z}_+$, $\lambda_1, \dots, \lambda_k > 0$ and $\emptyset \subset T_1 \subset T_2 \subset \dots \subset T_k \subseteq S$ such that $x = \sum_{i=1}^k \lambda_i \chi^{T_i}$, where χ^{T_i} is the incidence vector of T_i. Then $f'(x) := \sum_{i=1}^k \lambda_i f(T_i)$. Prove that f is submodular if and only if f' is convex.

(5 points)

Exercise 11.4. Let $f\colon 2^U \to \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$. Prove that the set of vertices of the base polyhedron of f is precisely the set of vectors $b^\prec$ for all total orders $\prec$ of U, where

$$b^\prec(u) := f\Big(\{v \in U : v \preceq u\}\Big) - f\Big(\{v \in U : v \prec u\}\Big) \qquad (u \in U).$$

(5 points)

Information: Submissions in groups of up to two students are allowed.

Professor Dr. Jens Vygen
Rudolf Scheifele

Deadline: Tuesday, January 15, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws18/coex.html`

In case of any questions feel free to contact me at scheifele@or.uni-bonn.de.