Exercise Set 11

Exercise 11.1. Let \((G, u, s, t)\) be a network and \(U := \delta^+(s)\). Let

\[ P := \{ x \in \mathbb{R}_+^U : \text{there is an } s-t \text{ flow } f \text{ in } (G, u) \text{ with } f(e) = x_e \text{ for all } e \in U \}. \]

Prove that \(P\) is a polymatroid.

(5 points)

Exercise 11.2. Prove that a nonempty compact set \(P \subseteq \mathbb{R}_+^n\) is a polymatroid if and only if

(i) For all \(0 \leq x \leq y \in P\) we have \(x \in P\).

(ii) For all \(x \in \mathbb{R}_+^n\) and all \(y, z \leq x\) with \(y, z \in P\) that are maximal with this property (i.e. \(y \leq w \leq x\) and \(w \in P\) implies \(w = y\), and \(z \leq w \leq x\) and \(w \in P\) implies \(w = z\)) we have \(1y = 1z\), where \(1\) is the vector whose entries are all 1.

(5 points)

Exercise 11.3. Let \(S\) be a finite set and \(f: 2^S \to \mathbb{R}\) with \(f(\emptyset) = 0\). Define \(f': \mathbb{R}_+^S \to \mathbb{R}\) as follows: For any \(x \in \mathbb{R}_+^S\) there are unique \(k \in \mathbb{Z}_+, \lambda_1, \ldots, \lambda_k > 0\) and \(\emptyset \subset T_1 \subset T_2 \subset \ldots \subset T_k \subseteq S\) such that \(x = \sum_{i=1}^k \lambda_i \chi_{T_i}\), where \(\chi_{T_i}\) is the incidence vector of \(T_i\). Then \(f'(x) := \sum_{i=1}^k \lambda_i f(T_i)\). Prove that \(f\) is submodular if and only if \(f'\) is convex.

(5 points)

Exercise 11.4. Let \(f: 2^U \to \mathbb{R}\) be a submodular function with \(f(\emptyset) = 0\). Prove that the set of vertices of the base polyhedron of \(f\) is precisely the set of vectors \(b^\prec\) for all total orders \(\prec\) of \(U\), where

\[ b^\prec(u) := f\left(\{v \in U : v \preceq u\}\right) - f\left(\{v \in U : v \prec u\}\right) \quad (u \in U). \]

(5 points)

Information: Submissions in groups of up to two students are allowed.
Deadline: Tuesday, January 15, before the lecture. The websites for lecture and exercises can be found at:

http://www.or.uni-bonn.de/lectures/ws18/coex.html

In case of any questions feel free to contact me at scheifele@or.uni-bonn.de.